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THE SYMMETRIC JOIN OF A COMPLEX

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1. **The definition of J .** Let K be a polyhedron. With each pair of distinct points p, q of K we associate a closed line segment pq . No distinction is made between p and q and the corresponding end points of pq . The length of pq is a continuous function of p and q , and the length approaches zero if p and q approach a common limit. Distinct segments do not intersect except at a common end point. The points of these segments with their obvious natural topology make up J , the symmetric join of K . This space arises in [4]¹ in connection with the problem of finding the chords of a manifold that are orthogonal to the manifold.

2. **The subdivision of J .** Let the mid-point of pq be denoted by $\Delta p \times q = \Delta q \times p$, and let $p = \Delta p \times p$. These points $\Delta p \times q$ make up the symmetric product S of K . Let the mid-point of the segment from p to $\Delta p \times q$ be denoted by $p \times q$, and let $p = p \times p$. These points $p \times q$ make up the topological product $P = K \times K$. Consider the closed segment of pq from $p \times q$ to $q \times p$, it being understood that this segment is the point p when $p = q$. All such segments form the "neighborhood" N_S . Clearly N_S can be homotopically deformed in N_S along the segments pq upon S with S remaining pointwise invariant. Finally consider the closed segment of pq from p to $p \times q$, it being understood that this segment is the point p when $p = q$. All such segments form the "neighborhood" N_K . Clearly N_K can be homotopically deformed in N_K along the segments pq upon K with K remaining pointwise invariant.

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¹ Numbers in brackets refer to the References at the end of the paper.

There is a natural cell division of P whose oriented cells are $\sigma \times \tau$ for all oriented simplexes σ, τ of K .² Let $|\sigma \times \tau|$ denote the carrier of $\sigma \times \tau$, that is, the point set of J associated with $\sigma \times \tau$. The rays pq determine a mapping Λ of P on S . The cells $\Lambda|\sigma \times \tau| = \Lambda|\tau \times \sigma|$ form a natural cell division of S .³ With this division Λ is a cell mapping. Furthermore $\Lambda\sigma \times \tau$ is an orientation of $\Lambda|\sigma \times \tau|$ such that

$$(1) \quad \Lambda\sigma^s \times \tau^t = (-1)^{st} \Lambda\tau^t \times \sigma^s,$$

the superscripts denoting dimension.³

THEOREM 1. *There is a simplicial division of J such that K, S, P, N_S , and N_K carry subcomplexes of the division, the first of which is a subdivision of the given polyhedron K , and the second and third of which are subdivisions of the above natural cell divisions of S and P .*

PROOF. Let $[\sigma, \tau]$ be the point set made up of the points of all segments pq with $p \in \sigma$ and $q \in \tau$. If σ and τ have no common vertex, $[\sigma, \tau] = (\sigma, \tau)$, the join⁴ of σ and τ . In this case S and P separate $[\sigma, \tau]$ into four $(s+t+1)$ -cells, where s and t are the dimensions of σ and τ , respectively.

Consider $[\sigma, \sigma]$. As an auxiliary set we construct the join (σ, σ') with σ' homeomorphic to σ . Each pair of points, one in σ and one in σ' , determines their join in (σ, σ') . This join of the two points is called a ray of (σ, σ') . The mid-points of the rays form a set homeomorphic to $\sigma \times \sigma'$ or $\sigma \times \sigma$. This product separates (σ, σ') into two cells. We discard the one of these cells that contains σ' and retain C , the closure of the other cell which contains $\sigma, \sigma \times \sigma$, and rays connecting these two sets. We subdivide $\sigma \times \sigma$ into a simplicial complex in such a way that the set of points $p \times p$, all p in σ , carries a subcomplex (that such a division exists is proved in [5]). Each simplex of this division is extended to a cell of C by adding to the simplex the points of all rays of C with one end point in the simplex (the set so obtained is a cell because it is obtained from a prism whose bases are simplexes by simple identifications in one base of the prism). These cells form a division of C into a cell complex. The rays of C that join p and $p \times p$, all p in σ , carry a subcomplex of this cell division. Let each such ray be reduced to a point by identification of all points of the ray. It is seen that the transform of C under this identification is a cell complex which is a division of $[\sigma, \sigma] \cap N_K$.

² The properties of P which we use can be found in [3].

³ See [5].

⁴ A discussion of the join of two complexes is found in [3]. Also the properties of the join which we assume are presented in [1].

To obtain a division of $[\sigma, \sigma] \cap N_S$ we consider the prism D defined as the product of $\sigma \times \sigma$ and a 1-simplex. We use the same simplicial division of $\sigma \times \sigma$ used in the preceding paragraph. This division is extended to a cell division of D by adding to each simplex the product of the simplex and the 1-simplex used to define D . One base of the prism D is reduced to the symmetric product of σ by the identification of $p \times q$ and $q \times p$ for all p, q in σ . If our original simplicial division of $\sigma \times \sigma$ is properly determined (as in [5]), these identifications transform D into a cell complex E .⁸ Next the rays of E which are images of the rays of D with end points $p \times p$, all p in σ , are reduced each to a point by identification. It is seen that the resulting space is a cell complex which is a division of $[\sigma, \sigma] \cap N_S$.

In the same way we subdivide $[\sigma, \tau]$ where σ and τ have some but not all of their vertices in common. It is seen that Theorem 1 is true.

3. Homologies in P and S . Let $z_i, i=1, 2, \dots$, be the cycles of a canonical basis⁵ for the chains of all dimensions of K , and let f_i be the non-cyclic chains of this canonical basis. The range of the subscript of f is a subset of the range of the subscript of z . For each f there is a boundary relation $Ff_i^{r+1} = e_i z_i^s, e_i \geq 1$. With each z_i that is free there is associated an e_i equal to zero.

A homology base for P is given by the cycles $z_i \times z_j$ and $(1/(e_i, e_j))F(f_i \times f_j)$, the denominator of the coefficient denoting the greatest common divisor of e_i and e_j . A complete set of homology relations for the cycles of this base is given by the two sets

$$(2) \quad (e_i, e_j)z_i \times z_j \sim 0, \quad F(f_i \times f_j) \sim 0,$$

with the understanding that $(a, 0) = a$.

We next obtain a homology base for S . We consider

$$(3) \quad \Lambda z_i^r \times z_j^s = (-1)^{rs} \Lambda z_j^s \times z_i^r, \quad i \neq j,$$

the equality holding because of (1). The dimension superscripts in (3) could be r_i and r_j rather than r and s , but we use the less clear notation to simplify the printing. We also consider

$$(4) \quad \Lambda z_i^r \times z_i^r, \quad r \text{ even,}$$

$$(5) \quad \Lambda \frac{1}{(e_i, e_j)} F(f_i^{r+1} \times f_j^{s+1}) = (-1)^{(r+1)(s+1)} \Lambda \frac{1}{(e_i, e_j)} F(f_j^{s+1} \times f_i^{r+1}),$$

$i \neq j,$

⁵ See [3, p. 104].

and

$$(6) \quad \frac{1}{2} \Lambda \frac{1}{e_i} F(j_i^{r+1} \times j_i^{r+1}), \quad r \text{ odd.}$$

Using [6, p. 22, line 15] and the method of [2, §4] we infer that the cycles (6) are integral, the orders of (3), (4), and (5) are (e_i, e_j) while (6) have the orders $2e_i$, the cycles (3), (4), (5), and (6) are a homology base for S , and the orders of these cycles are a complete set of homology relations in S .

4. The sum cycles and their group S . We shall apply the theory of the sum of two complexes to $J = N_S + N_K$. To prepare for this we define a sum cycle to be a cycle that is expressible as the sum of two cycles, one in N_S and the other in N_K . The group of the sum cycles modulo those that bound in J is called S . This group S is known to be a subgroup of the Betti group of J . Here, as throughout the paper unless otherwise stated, we do not define a group for each dimension but combine all elements of all dimensions into a single group.

THEOREM 2. *The group S is generated by a free 0-dimensional element and a set of elements of order 2, each containing one and only one of the cycles (6).*

PROOF. Any cycle of K can be deformed in J onto a point p of K along the segments pq . Since N_K is a deformation retract of K , the group of the cycles of N_K modulo those that bound in J has a single nonzero 0-dimensional element as generator.

A cycle of S which bounds in J is homologous in N_S to a cycle of P . Indeed if $z = FC$, z in S , C in J , and C simplicial, then C can be expressed as the sum of two chains, one in N_S and the other in N_K . The boundary of the first of these two chains is z minus a cycle of P .

Any cycle z of P is homologous in N_S to Λz . Hence using (3), (4), (5), and (6) we obtain Theorem 2.

5. The seams and their group Γ . The seams are defined to be the cycles of P that bound both in N_S and N_K . We note that if z is a cycle of P and $z \sim 0$ in N_S , then since z and Λz are homotopic, $\Lambda z \sim 0$ in N_S . But N_S was shown in §1 to be a neighborhood retract of S . Hence $\Lambda z \sim 0$ in S . From this observation, from (1), and from §3 we deduce that the cycles of P that bound in N_S are the cycles homologous in P to all linear combinations of

$$(7) \quad (z_i^r \times z_j^s) - (-1)^{rs} (z_j^s \times z_i^r), \quad i \neq j,$$

$$(8) \quad z_i^r \times z_j^r, \quad r \text{ odd,}$$

$$(9) \quad \frac{1}{(e_i, e_j)} [F(f_i^{r+1} \times f_j^{s+1}) - (-1)^{(r+1)(s+1)} F(f_j^{s+1} \times f_i^{r+1})], \quad i \neq j,$$

and

$$(10) \quad \frac{1}{e_i} F(f_i^{r+1} \times f_i^{r+1}), \quad r \text{ even.}$$

To find the cycles of P that bound in N_K we consider the transformation $M(p \times q) = p$. We have $M|\sigma \times \tau| = |\sigma|$, the notation being that of §2. Also for any set $A \subset P$ the discussion of §2 implies that A and $M(A)$ are homotopic in N_K . Hence $z_i^r \times z_j^s$ is homologous in N_K to $M(z_i^r \times z_j^s)$ which is carried by $|z_i^r|$. Hence if $s > 0$, $z_i^r \times z_j^s$ bounds in N_K because it can be homotopically deformed into a set of dimension less than the dimension of the cycle.

Suppose $s = 0$. We assume that z_i^0 is equal to 1 at exactly one vertex of K and is equal to 0 at all other vertices. Then $M(z_i^r \times z_j^0) = z_i^r$ because $z_i^r \times z_j^0$ is obtained by sliding z_i^r along the rays of a cone.

If in f_j^{s+1} the exponent s is greater than 0, $(1/(e_i, e_j))F(f_i^{r+1} \times f_j^{s+1}) \sim 0$ in N_K because it is homotopic to a cycle carried by $|f_i^{r+1}|$, and the dimension of the cycle is $r+s+1$ which is greater than $r+1$. If $s = 0$, the same homology holds because $e_j = 1$.

We have shown that the cycles of P that bound in N_K are the cycles homologous in P to all linear combinations of

$$z_i^r \times z_j^s, \quad s > 0,$$

$$z_i^r \times z_1^0 - z_i^r \times z_j^0, \quad i \neq 1,$$

and

$$\frac{1}{(e_i, e_j)} F(f_i^{r+1} \times f_i^{s+1}).$$

It is easily seen that if one cycle of a homology class of the Betti group of P is a seam, then all cycles of the class are seams. Furthermore such classes form a subgroup of the Betti group. This subgroup of the Betti group of P made up of the homology classes whose cycles are seams is called Γ . Combining the results of this section we obtain the following theorem.

THEOREM 3. *The subgroup Γ is generated by the homology classes that contain (8), (9), (10),*

$$(11) \quad (7) \text{ with } r > 0 \text{ and } s > 0,$$

and

$$(12) \quad (z_i^r \times z_1^0) - (z_i^r \times z_j^0) - (z_1^0 \times z_i^r) + (z_j^0 \times z_i^r), \quad i \neq 1.$$

6. The Betti group $B(J)$ of J . The theory of the sum of two complexes⁶ gives the isomorphism $B(J)/S \cong \Gamma$. However we shall obtain a sharper result.

For each seam z we shall define a pair of singular chains c of N_S and d of N_K such that $Fc = Fd = z$. To define c we deform z along the segments pq from P to S sweeping out the continuous image of a prism whose bases are $|z|$ and $|\Lambda z|$. This deformation gives rise to a singular chain c whose boundary⁷ is $z - \Lambda z$. But since z is a seam it follows from (1), (8), (9), (10), (11), and (12) that $\Lambda z = 0$. In a similar way we define the chain d of N_K .

Let $H(z)$ be the homology class of $B(J)$ which contains $c - d$. We know that any element of the Betti group $B(J)$ is expressible as the sum of an element of S and an $H(z)$, z a seam.⁶ Hence to characterize $B(J)$ we need only find the relations that involve the $H(z)$. We know⁶ that any homology among the $H(z)$ implies the same homology in P among the z . On the other hand, $z \sim 0$ in P implies⁶ that $H(z)$ is in S . We shall examine all relations $z \sim 0$ in P , and find the corresponding relations among the $H(z)$. The relations to examine are (2) applied to (8), (9), (10), (11), and (12).

First we consider $(e_i, e_j)(11) \sim 0$ in P . We consider

$$\begin{aligned}
 13) \quad C = & \alpha [(f_i^{r+1} \times z_j^s) - (-1)^{s(r+1)} (z_j^s \times f_i^{r+1})] \\
 & + \beta [(-1)^r (z_i^r \times f_j^{s+1}) - (-1)^{rs} (f_j^{s+1} \times z_i^r)],
 \end{aligned}$$

with α, β integers such that $\alpha e_i + \beta e_j = (e_i, e_j)$. Since we find in [3, p. 138, (5.5)] that

$$14) \quad F(x^p \times y) = (F x^p \times y) + (-1)^p (x^p \times F y),$$

it follows that $FC = (e_i, e_j)(11)$. Next we deform C along the segments pq from P to S sweeping out the continuous image of a prism whose bases are C and ΛC . This deformation gives rise to a singular chain⁷ whose boundary is $\Lambda C - C - (e_i, e_j)c$ because, as defined above, $(e_i, e_j)c$ is swept out by $FC = (e_i, e_j)z$. Using (1) we calculate that $\Lambda C = 0$. Hence $(e_i, e_j)c + C$ bounds in N_S . In the same way it is seen that $(e_i, e_j)d + C$ bounds in N_K . Hence $(e_i, e_j)(c - d) \sim 0$ in J and $(e_i, e_j)H(z) = 0$.

In the same way we show that $e_i H(12) = (e_i, e_j)H(9) = e_i H(10) = 0$. The argument is as above with (13) replaced by

⁶ See [3, p. 267, 18].

⁷ See [3, p. 307, (21.2)].

$$C = (f_i^{r+1} \times z_1^0) - (f_i^{r+1} \times z_j^0) - (z_1^0 \times f_i^{r+1}) + (z_j^0 \times f_i^{r+1}),$$

$$C = (f_i^{r+1} \times f_j^{s+1}) - (-1)^{(r+1)(s+1)}(f_j^{s+1} \times f_i^{r+1}),$$

and

$$C = (f_i^{r+1} \times f_i^{r+1}),$$

respectively.

Finally we consider $e_i(z_i^r \times z_i^r) \sim 0$ in P , r odd. Take $C = (f_i^{r+1} \times z_i^r)$. Then $FC = e_i(z_i^r \times z_i^r)$. Using (14), (1), and the fact that r is odd, we compute that

$$\Delta F(f_i^{r+1} \times z_i^r) = 2\Delta(f_i^{r+1} \times z_i^r).$$

Hence

$$e_i H(z_i^r \times z_i^r) \supset (1/2)\Delta F(f_i^{r+1} \times z_i^r).$$

This implies that the elements of S with dimension greater than zero are in the subgroup of $B(J)$ generated by the $H(z)$, z a seam. Also $H(z_i^r \times z_i^r)$, r odd, is of order $2e_i$. Our discussion implies the following:

THEOREM 4. *The Betti group $B(J)$ is generated by a nonzero 0-dimensional element, $H(8)$, $H(9)$, $H(10)$, $H(11)$, and $H(12)$; furthermore a complete set of relations for these generators consists of $2e_i H(8) = (e_i, e_j)H(9) = e_i H(10) = (e_i, e_j)H(11) = e_i H(12) = 0$.*

We shall conclude with two corollaries of Theorem 4. Let K_a be K augmented as defined in [3, p. 130]. Let $B(K_a)$ be generated by U_i , $i = 1, 2, \dots$, and u_k , $k = 1, 2, \dots$, where the U_i are free and independent, u_k is of order e_k , and the e_k are the torsion coefficients. Let the dimensions of U_i and u_k be r_i and r_k , respectively. Let B^n denote the n -dimensional Betti group of J .

COROLLARY 1. *To each pair U_i and U_j , $i \neq j$, $r_i + r_j = n - 1$, there corresponds a free element of B^n ; to each U_i , $r_i = (n - 1)/2$, r_i odd, there corresponds a free element of B^n ; to each pair u_k and u_l , $k \neq l$, $r_k + r_l =$ either $n - 1$ or $n - 2$, there corresponds an element of B^n of order (e_k, e_l) ; to each u_k , $r_k = (n - 1)/2$, r_k odd, there corresponds an element of B^n of order $2e_k$; and to each u_k , $r_k = (n - 2)/2$, r_k even, there corresponds an element of B^n of order e_k ; the elements mentioned generate B^n , and their orders are a complete set of relations for these generators; finally B^0 is a free cyclic group of rank 1.*

Let R^n and $R^n(J)$ denote the n -dimensional Betti numbers of K_a and J , respectively.

COROLLARY 2. *We have*

$$R^n(J) = \sum_i R^i R^{n-i-1} + A^n,$$

with the summation over all integers i from 0 to the greatest integer less than $(n-1)/2$ and

$$A^n = \begin{cases} R^{(n-1)/2}, & (n-1)/2 \text{ an odd integer,} \\ 0, & (n-1)/2 \text{ not an odd integer.} \end{cases}$$

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