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THE SYMMETRIC JOIN OF A COMPLEX

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1. **The definition of J.** Let K be a polyhedron. With each pair of distinct points p, q of K we associate a closed line segment pq. No distinction is made between p and q and the corresponding end points of pq. The length of pq is a continuous function of p and q, and the length approaches zero if p and q approach a common limit. Distinct segments do not intersect except at a common end point. The points of these segments with their obvious natural topology make up J, the symmetric join of K. This space arises in [4][1] in connection with the problem of finding the chords of a manifold that are orthogonal to the manifold.

2. **The subdivision of J.** Let the mid-point of pq be denoted by $\Lambda p\times q=\Lambda q\times p$, and let $p=\Lambda p\times p$. These points $\Lambda p\times q$ make up the symmetric product S of K. Let the mid-point of the segment from p to $\Lambda p\times q$ be denoted by $p\times q$, and let $p=p\times p$. These points $p\times q$ make up the topological product $P=K\times K$. Consider the closed segment of pq from $p\times q$ to $q\times p$, it being understood that this segment is the point p when $p=q$. All such segments form the "neighborhood" N_S. Clearly N_S can be homotopically deformed in N_S along the segments pq upon S with S remaining pointwise invariant. Finally consider the closed segment of pq from p to $p\times q$, it being understood that this segment is the point p when $p=q$. All such segments form the "neighborhood" N_K. Clearly N_K can be homotopically deformed in N_K along the segments pq upon K with K remaining pointwise invariant.

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[1] Numbers in brackets refer to the References at the end of the paper.

There is a natural cell division of P whose oriented cells are $\sigma\times\tau$ for all oriented simplexes σ, τ of K.[2] Let $|\sigma\times\tau|$ denote the carrier of $\sigma\times\tau$, that is, the point set of J associated with $\sigma\times\tau$. The rays pq determine a mapping Λ of P on S. The cells $\Lambda|\sigma\times\tau|=\Lambda|\tau\times\sigma|$ form a natural cell division of S.[3] With this division Λ is a cell mapping. Furthermore $\Lambda\sigma\times\tau$ is an orientation of $\Lambda|\sigma\times\tau|$ such that

$$\Lambda\sigma^s \times \tau^t = (-1)^{st}\Lambda\tau^t \times \sigma^s, \tag{1}$$

the superscripts denoting dimension.[3]

THEOREM 1. *There is a simplicial division of J such that K, S, P, N_S, and N_K carry subcomplexes of the division, the first of which is a subdivision of the given polyhedron K, and the second and third of which are subdivisions of the above natural cell divisions of S and P.*

PROOF. Let $[\sigma, \tau]$ be the point set made up of the points of all segments pq with $p\in\sigma$ and $q\in\tau$. If σ and τ have no common vertex, $[\sigma,\tau]=(\sigma,\tau)$, the join[4] of σ and τ. In this case S and P separate $[\sigma,\tau]$ into four $(s+t+1)$-cells, where s and t are the dimensions of σ and τ, respectively.

Consider $[\sigma, \sigma]$. As an auxiliary set we construct the join (σ, σ') with σ' homeomorphic to σ. Each pair of points, one in σ and one in σ', determines their join in (σ, σ'). This join of the two points is called a ray of (σ, σ'). The mid-points of the rays form a set homeomorphic to $\sigma\times\sigma'$ or $\sigma\times\sigma$. This product separates (σ,σ') into two cells. We discard the one of these cells that contains σ' and retain C, the closure of the other cell which contains σ, $\sigma\times\sigma$, and rays connecting these two sets. We subdivide $\sigma\times\sigma$ into a simplicial complex in such a way that the set of points $p\times p$, all p in σ, carries a subcomplex (that such a division exists is proved in [5]). Each simplex of this division is extended to a cell of C by adding to the simplex the points of all rays of C with one end point in the simplex (the set so obtained is a cell because it is obtained from a prism whose bases are simplexes by simple identifications in one base of the prism). These cells form a division of C into a cell complex. The rays of C that join p and $p\times p$, all p in σ, carry a subcomplex of this cell division. Let each such ray be reduced to a point by identification of all points of the ray. It is seen that the transform of C under this identification is a cell complex which is a division of $[\sigma,\sigma]\cap N_K$.

[2] The properties of P which we use can be found in [3].

[3] See [5].

[4] A discussion of the join of two complexes is found in [3]. Also the properties of the join which we assume are presented in [1].

To obtain a division of $[\sigma, \sigma] \cap N_S$ we consider the prism D defined as the product of $\sigma \times \sigma$ and a 1-simplex. We use the same simplicial division of $\sigma \times \sigma$ used in the preceding paragraph. This division is extended to a cell division of D by adding to each simplex the product of the simplex and the 1-simplex used to define D. One base of the prism D is reduced to the symmetric product of σ by the identification of $p \times q$ and $q \times p$ for all p, q in σ. If our original simplicial division of $\sigma \times \sigma$ is properly determined (as in [5]), these identifications transform D into a cell complex E.[3] Next the rays of E which are images of the rays of D with end points $p \times p$, all p in σ, are reduced each to a point by identification. It is seen that the resulting space is a cell complex which is a division of $[\sigma, \sigma] \cap N_S$.

In the same way we subdivide $[\sigma, \tau]$ where σ and τ have some but not all of their vertices in common. It is seen that Theorem 1 is true.

3. **Homologies in P and S.** Let z_i, $i=1, 2, \cdots$, be the cycles of a canonical basis[5] for the chains of all dimensions of K, and let f_i be the non-cyclic chains of this canonical basis. The range of the subscript of f is a subset of the range of the subscript of z. For each f there is a boundary relation $Ff_i^{r+1} = e_i z_i^r$, $e_i \geqq 1$. With each z_i that is free there is associated an e_i equal to zero.

A homology base for P is given by the cycles $z_i \times z_j$ and $(1/(e_i, e_j))F(f_i \times f_j)$, the denominator of the coefficient denoting the greatest common divisor of e_i and e_j. A complete set of homology relations for the cycles of this base is given by the two sets

$$(e_i, e_j) z_i \times z_j \sim 0, \qquad F(f_i \times f_j) \sim 0, \tag{2}$$

with the understanding that $(a, 0) = a$.

We next obtain a homology base for S. We consider

$$\Lambda \overset{r}{z}_i \times \overset{s}{z}_j = (-1)^{rs} \Lambda \overset{s}{z}_j \times \overset{r}{z}_i, \qquad i \neq j, \tag{3}$$

the equality holding because of (1). The dimension superscripts in (3) could be r_i and r_j rather than r and s, but we use the less clear notation to simplify the printing. We also consider

$$\Lambda \overset{r}{z}_i \times \overset{r}{z}_i, \qquad r \text{ even}, \tag{4}$$

$$\Lambda \frac{1}{(e_i, e_j)} F(f_i^{r+1} \times f_j^{s+1}) = (-1)^{(r+1)(s+1)} \Lambda \frac{1}{(e_i, e_j)} F(f_j^{s+1} \times f_i^{r+1}), \qquad i \neq j, \tag{5}$$

[5] See [3, p. 104].

and

$$\frac{1}{2}\,\Lambda\,\frac{1}{e_i}F(\overset{r+1}{f_i}\times\overset{r+1}{f_i}),\qquad r\text{ odd.}\tag{6}$$

Using [6, p. 22, line 15] and the method of [2, §4] we infer that the cycles (6) are integral, the orders of (3), (4), and (5) are (e_i, e_j) while (6) have the orders $2e_i$, the cycles (3), (4), (5), and (6) are a homology base for S, and the orders of these cycles are a complete set of homology relations in S.

4. **The sum cycles and their group** S. We shall apply the theory of the sum of two complexes to $J=N_S+N_K$. To prepare for this we define a sum cycle to be a cycle that is expressible as the sum of two cycles, one in N_S and the other in N_K. The group of the sum cycles modulo those that bound in J is called S. This group S is known to be a subgroup of the Betti group of J. Here, as throughout the paper unless otherwise stated, we do not define a group for each dimension but combine all elements of all dimensions into a single group.

THEOREM 2. *The group S is generated by a free* 0*-dimensional element and a set of elements of order* 2, *each containing one and only one of the cycles* (6).

PROOF. Any cycle of K can be deformed in J onto a point p of K along the segments pq. Since N_K is a deformation retract of K, the group of the cycles of N_K modulo those that bound in J has a single nonzero 0-dimensional element as generator.

A cycle of S which bounds in J is homologous in N_S to a cycle of P. Indeed if $z=FC$, z in S, C in J, and C simplicial, then C can be expressed as the sum of two chains, one in N_S and the other in N_K. The boundary of the first of these two chains is z minus a cycle of P.

Any cycle z of P is homologous in N_S to Λz. Hence using (3), (4), (5), and (6) we obtain Theorem 2.

5. **The seams and their group** Γ. The seams are defined to be the cycles of P that bound both in N_S and N_K. We note that if z is a cycle of P and $z\sim 0$ in N_S, then since z and Λz are homotopic, $\Lambda z\sim 0$ in N_S. But N_S was shown in §1 to be a neighborhood retract of S. Hence $\Lambda z\sim 0$ in S. From this observation, from (1), and from §3 we deduce that the cycles of P that bound in N_S are the cycles homologous in P to all linear combinations of

$$(\overset{r}{z_i}\times\overset{s}{z_j})-(-1)^{rs}(\overset{s}{z_j}\times\overset{r}{z_i}),\qquad i\neq j,\tag{7}$$

$$\overset{r}{z}_i \times \overset{r}{z}_i, \qquad r \text{ odd}, \tag{8}$$

$$\frac{1}{(e_i, e_j)}\left[F(\overset{r+1}{f_i} \times \overset{s+1}{f_j}) - (-1)^{(r+1)(s+1)} F(\overset{s+1}{f_j} \times \overset{r+1}{f_i})\right], \qquad i \neq j, \tag{9}$$

and

$$\frac{1}{e_i} F(\overset{r+1}{f_i} \times \overset{r+1}{f_i}), \qquad r \text{ even}. \tag{10}$$

To find the cycles of P that bound in N_K we consider the transformation $M(p \times q) = p$. We have $M|\sigma \times \tau| = |\sigma|$, the notation being that of §2. Also for any set $A \subset P$ the discussion of §2 implies that A and $M(A)$ are homotopic in N_K. Hence $\overset{r}{z}_i \times \overset{s}{z}_j$ is homologous in N_K to $M(\overset{r}{z}_i \times \overset{s}{z}_j)$ which is carried by $|\overset{r}{z}_i|$. Hence if $s > 0$, $\overset{r}{z}_i \times \overset{s}{z}_j$ bounds in N_K because it can be homotopically deformed into a set of dimension less than the dimension of the cycle.

Suppose $s = 0$. We assume that $\overset{0}{z}_i$ is equal to 1 at exactly one vertex of K and is equal to 0 at all other vertices. Then $M(\overset{r}{z}_i \times \overset{0}{z}_j) = \overset{r}{z}_i$ because $\overset{r}{z}_i \times \overset{0}{z}_j$ is obtained by sliding $\overset{r}{z}_i$ along the rays of a cone.

If in f_j^{s+1} the exponent s is greater than 0, $(1/(e_i, e_j))F(f_i^{r+1} \times f_j^{s+1}) \sim 0$ in N_K because it is homotopic to a cycle carried by $|f_i^{r+1}|$, and the dimension of the cycle is $r+s+1$ which is greater than $r+1$. If $s = 0$, the same homology holds because $e_j = 1$.

We have shown that the cycles of P that bound in N_K are the cycles homologous in P to all linear combinations of

$$\overset{r}{z}_i \times \overset{s}{z}_j, \qquad s > 0,$$

$$\overset{r}{z}_i \times \overset{0}{z}_1 - \overset{r}{z}_i \times \overset{0}{z}_j, \qquad i \neq 1,$$

and

$$\frac{1}{(e_i, e_j)} F(\overset{r+1}{f_i} \times \overset{s+1}{f_j}).$$

It is easily seen that if one cycle of a homology class of the Betti group of P is a seam, then all cycles of the class are seams. Furthermore such classes form a subgroup of the Betti group. This subgroup of the Betti group of P made up of the homology classes whose cycles are seams is called Γ. Combining the results of this section we obtain the following theorem.

THEOREM 3. *The subgroup* Γ *is generated by the homology classes that contain* (8), (9), (10),

(11) (7) *with* $r > 0$ *and* $s > 0$,

and

$$(\overset{r}{z}_i \times \overset{0}{z}_1) - (\overset{r}{z}_i \times \overset{0}{z}_j) - (\overset{0}{z}_1 \times \overset{r}{z}_i) + (\overset{0}{z}_j \times \overset{r}{z}_i), \qquad i \neq 1. \tag{12}$$

6. **The Betti group $B(J)$ of J.** The theory of the sum of two complexes[6] gives the isomorphism $B(J)/S \cong \Gamma$. However we shall obtain a sharper result.

For each seam z we shall define a pair of singular chains c of N_S and d of N_K such that $Fc = Fd = z$. To define c we deform z along the segments pq from P to S sweeping out the continuous image of a prism whose bases are $|z|$ and $|\Lambda z|$. This deformation gives rise to a singular chain c whose boundary[7] is $z - \Lambda z$. But since z is a seam it follows from (1), (8), (9), (10), (11), and (12) that $\Lambda z = 0$. In a similar way we define the chain d of N_K.

Let $H(z)$ be the homology class of $B(J)$ which contains $c - d$. We know that any element of the Betti group $B(J)$ is expressible as the sum of an element of S and an $H(z)$, z a seam.[6] Hence to characterize $B(J)$ we need only find the relations that involve the $H(z)$. We know[6] that any homology among the $H(z)$ implies the same homology in P among the z. On the other hand, $z \sim 0$ in P implies[6] that $H(z)$ is in S. We shall examine all relations $z \sim 0$ in P, and find the corresponding relations among the $H(z)$. The relations to examine are (2) applied to (8), (9), (10), (11), and (12).

First we consider $(e_i, e_j)(11) \sim 0$ in P. We consider

$$\begin{aligned} C = {} & \alpha[(\overset{r+1}{f_i} \times \overset{s}{z_j}) - (-1)^{s(r+1)}(\overset{s}{z_j} \times \overset{r+1}{f_i})] \\ & + \beta[(-1)^r(\overset{r}{z_i} \times \overset{s+1}{f_j}) - (-1)^{rs}(\overset{s+1}{f_j} \times \overset{r}{z_i})], \end{aligned} \tag{13}$$

with α, β integers such that $\alpha e_i + \beta e_j = (e_i, e_j)$. Since we find in [**3**, p. 138, (5.5)] that

$$F(x^p \times y) = (Fx^p \times y) + (-1)^p(x^p \times Fy), \tag{14}$$

it follows that $FC = (e_i, e_j)(11)$. Next we deform C along the segments pq from P to S sweeping out the continuous image of a prism whose bases are C and ΛC. This deformation gives rise to a singular chain[7] whose boundary is $\Lambda C - C - (e_i, e_j)c$ because, as defined above, $(e_i, e_j)c$ is swept out by $FC = (e_i, e_j)z$. Using (1) we calculate that $\Lambda C = 0$. Hence $(e_i, e_j)c + C$ bounds in N_S. In the same way it is seen that $(e_i, e_j)d + C$ bounds in N_K. Hence $(e_i, e_j)(c - d) \sim 0$ in J and $(e_i, e_j)H(z) = 0$.

In the same way we show that $e_i H(12) = (e_i, e_j)H(9) = e_i H(10) = 0$. The argument is as above with (13) replaced by

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[6] See [**3**, p. 267, 18].

[7] See [**3**, p. 307, (21.2)].

$$C = (f_i^{r+1} \times z_1^0) - (f_i^{r+1} \times z_j^0) - (z_1^0 \times f_i^{r+1}) + (z_j^0 \times f_i^{r+1}),$$
$$C = (f_i^{r+1} \times f_j^{s+1}) - (-1)^{(r+1)(s+1)}(f_j^{s+1} \times f_i^{r+1}),$$

and

$$C = (f_i^{r+1} \times f_i^{r+1}),$$

respectively.

Finally we consider $e_i(z_i^r \times z_i^r) \sim 0$ in P, r odd. Take $C = (f_i^{r+1} \times z_i^r)$. Then $FC = e_i(z_i^r \times z_i^r)$. Using (14), (1), and the fact that r is odd, we compute that

$$\Lambda F(f_i^{r+1} \times f_i^{r+1}) = 2\Lambda(f_i^{r+1} \times z_i^r).$$

Hence

$$e_i H(z_i^r \times z_i^r) \supset (1/2)\Lambda F(f_i^{r+1} \times f_i^{r+1}).$$

This implies that the elements of S with dimension greater than zero are in the subgroup of $B(J)$ generated by the $H(z)$, z a seam. Also $H(z_i^r \times z_i^r)$, r odd, is of order $2e_i$. Our discussion implies the following:

THEOREM 4. *The Betti group $B(J)$ is generated by a nonzero 0-dimensional element, $H(8)$, $H(9)$, $H(10)$, $H(11)$, and $H(12)$; furthermore a complete set of relations for these generators consists of $2e_iH(8) = (e_i, e_j)H(9) = e_iH(10) = (e_i, e_j)H(11) = e_iH(12) = 0$.*

We shall conclude with two corollaries of Theorem 4. Let K_a be K augmented as defined in [3, p. 130]. Let $B(K_a)$ be generated by U_i, $i = 1, 2, \cdots$, and u_k, $k = 1, 2, \cdots$, where the U_i are free and independent, u_k is of order e_k, and the e_k are the torsion coefficients. Let the dimensions of U_i and u_k be r_i and r_k, respectively. Let B^n denote the n-dimensional Betti group of J.

COROLLARY 1. *To each pair U_i and U_j, $i \neq j$, $r_i + r_j = n-1$, there corresponds a free element of B^n; to each U_i, $r_i = (n-1)/2$, r_i odd, there corresponds a free element of B^n; to each pair u_k and u_l, $k \neq l$, $r_k + r_l$ = either $n-1$ or $n-2$, there corresponds an element of B^n of order (e_k, e_l); to each u_k, $r_k = (n-1)/2$, r_k odd, there corresponds an element of B^n of order $2e_k$; and to each u_k, $r_k = (n-2)/2$, r_k even, there corresponds an element of B^n of order e_k; the elements mentioned generate B^n, and their orders are a complete set of relations for these generators; finally B^0 is a free cyclic group of rank* 1.

Let R^n and $R^n(J)$ denote the n-dimensional Betti numbers of K_a and J, respectively.

COROLLARY 2. *We have*

$$R^n(J) = \sum_i R^i R^{n-i-1} + A^n,$$

with the summation over all integers i from 0 *to the greatest integer less than* $(n-1)/2$ *and*

$$A^n = \begin{cases} R^{(n-1)/2}, & (n-1)/2 \text{ an odd integer}, \\ 0, & (n-1)/2 \text{ not an odd integer}. \end{cases}$$

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