versely if the first row is multiplied by the inverse of c (mod p^k). This inverse exists, and the correspondence is one-to-one, because c is prime to p. This proves (3).

The sum of the probabilities $P_n(ap^\alpha, p^k)$, where a runs through the values $1, 2, \cdots, p^{k-\alpha}$, is clearly the probability that a determinant be divisible by p^α. The terms of this sum can be simplified and collected by use of (3), and we have

$$P_n(0, p^\alpha) = \sum_{r=0}^{k-\alpha} \phi(p^{k-\alpha-r}) P_n(p^{\alpha+r}, p^k). \tag{11}$$

Replacing α by $\alpha+1$, and subtracting the resulting equation from (11), we arrive at (4).

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ON THE NOTION OF THE RING OF QUOTIENTS OF A PRIME IDEAL

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Let $\mathfrak{o}$ be a domain of integrity (that is, a ring with unit element and with no zero divisor not equal to 0), and let $\mathfrak{u}$ be a prime ideal in $\mathfrak{o}$. We can construct two auxiliary rings associated with $\mathfrak{u}$: the factor ring $\mathfrak{o}/\mathfrak{u}$, composed of the residue classes of elements of $\mathfrak{o}$ modulo $\mathfrak{u}$, and the ring of quotients $\mathfrak{o}_{\mathfrak{u}}$, composed of the fractions whose numerator and denominator belong to $\mathfrak{o}$, but whose denominators do not belong to $\mathfrak{u}$. These constructions are of paramount importance in algebraic geometry; if $\mathfrak{o}$ is the ring of a variety $\boldsymbol{V}$, there corresponds to $\mathfrak{u}$ a subvariety $\boldsymbol{U}$ of $\boldsymbol{V}$; $\mathfrak{o}/\mathfrak{u}$ is the ring of $\boldsymbol{U}$, whereas the ring $\mathfrak{o}_{\mathfrak{u}}$ is the proper algebraic tool to investigate the neighborhood of $\boldsymbol{U}$ with respect to $\boldsymbol{V}$.

Now, the local theory of algebraic varieties involves the consideration of rings which are not domains of integrity (this, because the completion of a local ring may introduce zero divisors). Let then $\mathfrak{o}$ be any commutative ring with unit element, and let again $\mathfrak{u}$ be a prime ideal in $\mathfrak{o}$. We may define the factor ring $\mathfrak{o}/\mathfrak{u}$ exactly in the same way as above, but we cannot so easily generalize the notion of the ring of quotients $\mathfrak{o}_{\mathfrak{u}}$. If there exist zero divisors outside $\mathfrak{u}$, these zero divisors cannot be used as denominators of fractions, which shows that the definition of $\mathfrak{o}_{\mathfrak{u}}$ cannot be extended verbatim. If we

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consider only fractions whose denominators are not zero divisors and do not belong to $\mathfrak{u}$, we obtain a ring $\mathfrak{o}'$; but $\mathfrak{o}'$ fails in general to have the essential property of a ring of quotients, namely, of being a local ring in the sense of Krull (that is, the non-units in $\mathfrak{o}'$ will not form an ideal). The object of this note is to construct a ring for which the essential properties of rings of quotients are preserved.

Throughout this paper we shall denote by $\mathfrak{o}$ a Noetherian ring (that is, a ring in which the maximal condition for ideals is satisfied) with a unit element. Generalizing the problem of defining the ring of quotients of a prime ideal, we take any multiplicatively closed[1] subset S of $\mathfrak{o}$ which does not contain 0 (a set is said to be multiplicatively closed if the product of any two elements of the set belongs to the set; if we are concerned with a prime ideal $\mathfrak{u}$ in $\mathfrak{o}$, we take S to be the complement of $\mathfrak{u}$ in $\mathfrak{o}$). There exists at least one primary ideal which does not meet S (otherwise, 0 would belong to S as we see at once by representing the zero ideal as an intersection of primary ideals). We shall denote by $\mathfrak{s}$ the intersection of all primary ideals in $\mathfrak{o}$ which do not meet S.

PROPOSITION 1. *Let $\{0\} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_h$ be an irredundant representation of the zero ideal in $\mathfrak{o}$ as an intersection of primary ideals, and let $\mathfrak{p}_i (1 \leqq i \leqq h)$ be the associated prime ideal of $\mathfrak{q}_i$. Assume that $\mathfrak{p}_i \cap S = \emptyset$ for $i \leqq g$, but not for $i > g$. Then $\mathfrak{s} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_g$.*

It is clear that $\mathfrak{s} \subset \mathfrak{q}' = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_g$. Let $\mathfrak{v}$ be any primary ideal which does not meet S; we shall prove that $\mathfrak{q}' \subset \mathfrak{v}$. Let $\mathfrak{q}''$ be the ideal $\mathfrak{q}_{g+1} \cap \cdots \cap \mathfrak{q}_h$. We have $\{0\} = \mathfrak{q}' \cap \mathfrak{q}'' = \mathfrak{q}'\mathfrak{q}'' \subset \mathfrak{v}$, whence $\mathfrak{q}' \subset \mathfrak{v} : \mathfrak{q}''$. Let $\mathfrak{u}$ be the associated prime ideal of $\mathfrak{v}$; since $\mathfrak{v}$ contains some power of $\mathfrak{u}$, it follows from the multiplicatively closed character of S that $\mathfrak{u}$ does not meet S. If $i > g$, the ideal $\mathfrak{p}_i$ meets S and is therefore not contained in $\mathfrak{u}$. It follows[2] that $\mathfrak{v} : \mathfrak{q}'' = \mathfrak{v}$, whence $\mathfrak{q}' \subset \mathfrak{v}$. Proposition 1 is thereby proved.

LEMMA 1. *Let $\mathfrak{p}$ be a prime ideal in $\mathfrak{o}$, and let $\mathfrak{a}$ be an ideal contained in $\mathfrak{p}$. If $\mathfrak{q}$ is an ideal containing $\mathfrak{a}$, the statements "$\mathfrak{q}$ is primary for $\mathfrak{p}$" and "$\mathfrak{q}/\mathfrak{a}$ is primary for $\mathfrak{p}/\mathfrak{a}$" are equivalent.*

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[1] It was H. Grell who observed for the first time that, S being any multiplicatively closed set of nonzero divisors in a ring, it is possible to associate with S a ring of quotients, whose elements are the fractions whose denominators belong to S (Cf. H. Grell, *Beziehungen zwischen Ideale verschiedener Ringe*, Math. Ann. vol. 97 (1926) p. 510). For the properties of these rings of quotients, cf. Krull, *Idealtheorie* (Ergebnisse der Mathematik) or my paper *On the theory of local rings*, Ann. of Math. vol. 44 (1943) p. 690.

[2] Cf. van der Waerden, *Moderne Algebra*, vol. 2, chap. 12, p. 36.

Lemma 1 follows trivially from the definitions.

The zero ideal in $\mathfrak{o}/\mathfrak{s}$ is the intersection of the primary ideals $\mathfrak{q}_1/\mathfrak{s}, \cdots, \mathfrak{q}_g/\mathfrak{s}$ whose associated prime ideals are $\mathfrak{p}_1/\mathfrak{s}, \cdots, \mathfrak{p}_g/\mathfrak{s}$. Let S^* be the set of the residue classes modulo $\mathfrak{s}$ of the elements of S; then $S^* \cap (\mathfrak{p}_i/\mathfrak{s}) = \varnothing$ $(1 \leqq i \leqq g)$, which means that no element of S^* is a zero divisor in $\mathfrak{o}/\mathfrak{s}$.[3] We may therefore construct the ring of quotients $(\mathfrak{o}/\mathfrak{s})_{S^*}$ of S^* with respect to the ring $\mathfrak{o}/\mathfrak{s}$.

DEFINITION 1. *The ring $(\mathfrak{o}/\mathfrak{s})_{S^*}$ will be called the ring of quotients of S with respect to $\mathfrak{o}$. This ring will be denoted by $\mathfrak{o}_S$.*

This definition coincides with the usual one in the case where S does not contain any zero divisor. We shall now prove that the essential properties of rings of quotients in the usual sense still hold in our case.

If $\mathfrak{a}$ is an ideal in $\mathfrak{o}$, $(\mathfrak{a}+\mathfrak{s}/\mathfrak{s})\mathfrak{o}_S$ is an ideal in $\mathfrak{o}_S$ which we shall denote symbolically by $\mathfrak{a}\mathfrak{o}_S$ (in spite of the fact that $\mathfrak{o}$ is not in general a subring of $\mathfrak{o}_S$, so that we cannot multiply elements of $\mathfrak{o}$ by elements of $\mathfrak{o}_S$). If $\mathfrak{b}$ is any ideal in $\mathfrak{o}_S$, the set $\mathfrak{b}\cap(\mathfrak{o}/\mathfrak{s})$ may be written in the form $\mathfrak{a}/\mathfrak{s}$, where $\mathfrak{a}$ is an ideal in $\mathfrak{o}$ which contains $\mathfrak{s}$. We shall denote $\mathfrak{a}$ symbolically by $\mathfrak{b}\cap\mathfrak{o}$ (although $\mathfrak{b}\cap\mathfrak{o}$ is not a set theoretic intersection).

PROPOSITION 2. *If $\mathfrak{b}$ is any ideal in $\mathfrak{o}_S$, we have $\mathfrak{b} = (\mathfrak{b}\cap\mathfrak{o})\mathfrak{o}_S$.*

Since $\mathfrak{o}_S$ is a ring of quotients of $\mathfrak{o}/\mathfrak{s}$, we have $\mathfrak{b} = (\mathfrak{b}\cap(\mathfrak{o}/\mathfrak{s}))\mathfrak{o}_S$.[1] Proposition 2 follows immediately from this formula.

PROPOSITION 3. *Let $\mathfrak{p}$ be a prime ideal in $\mathfrak{o}$, and let $\mathfrak{q}$ be primary for $\mathfrak{p}$. If $\mathfrak{q}$ meets S, we have $\mathfrak{p}\mathfrak{o}_S = \mathfrak{q}\mathfrak{o}_S = \mathfrak{o}_S$. If $\mathfrak{p}$ does not meet S, $\mathfrak{p}\mathfrak{o}_S$ is prime, $\mathfrak{q}\mathfrak{o}_S$ is primary for $\mathfrak{p}\mathfrak{o}_S$ and $\mathfrak{p}\mathfrak{o}_S\cap\mathfrak{o} = \mathfrak{p}$, $\mathfrak{q}\mathfrak{o}_S\cap\mathfrak{o} = \mathfrak{q}$.*

If $\mathfrak{q}$ meets S, $\mathfrak{q}+\mathfrak{s}/\mathfrak{s}$ meets S^*, whence $\mathfrak{p}\mathfrak{o}_S = \mathfrak{q}\mathfrak{o}_S = \mathfrak{o}_S$. If $\mathfrak{p}$ does not meet S, the same holds for $\mathfrak{q}$, whence $\mathfrak{s} \subset \mathfrak{q} \subset \mathfrak{p}$. By Lemma 1, $\mathfrak{p}/\mathfrak{s}$ is prime and $\mathfrak{q}/\mathfrak{s}$ is primary for $\mathfrak{p}/\mathfrak{s}$. Furthermore, $\mathfrak{p}/\mathfrak{s}$ does not meet S^*. Proposition 3 follows therefore from the corresponding proposition which is known to hold for ordinary rings of quotients.[1] We see also that, if $\mathfrak{p}$ does not meet $\mathfrak{s}$, the formula $\mathfrak{q} \leftrightarrow \mathfrak{q}\mathfrak{o}_S$ establishes a one-to-one inclusion preserving correspondence between the primary ideals for $\mathfrak{p}$ in $\mathfrak{o}$ and the primary ideals for $\mathfrak{p}\mathfrak{o}_S$ in $\mathfrak{o}_S$.

PROPOSITION 4. *Let $\mathfrak{a} = \mathfrak{v}_1 \cap \cdots \cap \mathfrak{v}_r$ be an irredundant representation of an ideal $\mathfrak{a}$ in $\mathfrak{o}$ as an intersection of primary ideals. Let $\mathfrak{u}_i$ be the associated prime ideal of $\mathfrak{v}_i$, and assume that $\mathfrak{u}_i$ meets S for $i > s$ but not for $i \leqq s$. Then we have $\mathfrak{a}\mathfrak{o}_S = \mathfrak{v}_1\mathfrak{o}_S \cap \cdots \cap \mathfrak{v}_s\mathfrak{o}_S$, and this is an irredundant representation of $\mathfrak{a}\mathfrak{o}_S$ as intersection of primary ideals in $\mathfrak{o}_S$.*

[3] If $s^* \in S^*$, we have $\{0\} : s^*(\mathfrak{o}/\mathfrak{s}) = \{0\}$ by the result quoted in footnote 2.

It is obvious that $\mathfrak{a}\mathfrak{o}_S \subset \mathfrak{v}_1\mathfrak{o}_S \cap \cdots \cap \mathfrak{v}_s\mathfrak{o}_S$. Let conversely a be any element of $\mathfrak{v}_1\mathfrak{o}_S \cap \cdots \cap \mathfrak{v}_s\mathfrak{o}_S$. We know[3] that $\mathfrak{v}_1\mathfrak{o}_S \cap \cdots \cap \mathfrak{v}_s\mathfrak{o}_S$ is equal to $(\mathfrak{v}_1/\mathfrak{s} \cap \cdots \cap \mathfrak{v}_s/\mathfrak{s})\mathfrak{o}_S$; it follows that a may be written in the form b^*/c^*, where $b^* \in \mathfrak{v}_1/\mathfrak{s} \cap \cdots \cap \mathfrak{v}_s/\mathfrak{s}$ and $c^* \in S^*$. If $i > s$, the ideal $\mathfrak{u}_i$ has an element u_i in common with S; if m is large enough, we have $u = (\prod_{i=s+1}^{r} u_i)^m \in \mathfrak{v}_{s+1} \cap \cdots \cap \mathfrak{v}_r$. Since $u_i \in S$ $(s+1 \leqq i \leqq r)$, we have $u \in S$, whence $u^* \in S^*$, if u^* is the residue class of u modulo $\mathfrak{s}$. We may write $a = (u^*b^*)/(u^*c^*)$, $u^*c^* \in S^*$. Let b be any element of the residue class b^* modulo $\mathfrak{s}$; since $b^* \in \mathfrak{v}_i/\mathfrak{s}$ $(1 \leqq i \leqq s)$, we have $b \in \mathfrak{v}_1 \cap \cdots \cap \mathfrak{v}_s$, whence $ub \in \mathfrak{a}$ and $u^*b^* \in \mathfrak{a}+\mathfrak{s}/\mathfrak{s}$, $a \in \mathfrak{a}\mathfrak{o}_S$. It is clear that none of the ideals $\mathfrak{v}_i/\mathfrak{s}$ contains the intersection of the others; making use of a known result[1] for ordinary rings of quotients, it follows that the representation $\mathfrak{a}\mathfrak{o}_S = \mathfrak{v}_1\mathfrak{o}_S \cap \cdots \cap \mathfrak{v}_s\mathfrak{o}_S$ is irredundant.

We shall now consider more specifically the case where S is the complement of a prime ideal $\mathfrak{u}$. The ring $\mathfrak{o}_S$ will then also be denoted by $\mathfrak{o}_\mathfrak{u}$. In that case, *the ideal* $\mathfrak{s}$ *coincides with the intersection of all primary ideals for* $\mathfrak{u}$. In fact, the set S^* is clearly the complement of $\mathfrak{u}/\mathfrak{s}$ with respect to $\mathfrak{o}/\mathfrak{s}$; the ring $\mathfrak{o}_S$ is the ring of quotients (in the ordinary sense) of the prime ideal $\mathfrak{u}/\mathfrak{s}$ with respect to $\mathfrak{o}/\mathfrak{s}$. It follows that $\mathfrak{u}\mathfrak{o}_S$ is the ideal of non-units in $\mathfrak{o}_S$, whence $\bigcap_{n=1}^{\infty}(\mathfrak{u}/\mathfrak{s})^n\mathfrak{o}_S = \{0\}$.[4] For every n, the ideal $(\mathfrak{u}/\mathfrak{s})^n\mathfrak{o}_S \cap \mathfrak{o}/\mathfrak{s}$ is a primary ideal for $\mathfrak{u}/\mathfrak{s}$ in $\mathfrak{o}/\mathfrak{s}$; it follows that the intersection of all primary ideals for $\mathfrak{u}/\mathfrak{s}$ is the zero ideal in $\mathfrak{o}/\mathfrak{s}$. Our assertion then follows from Lemma 1.[5] At the same time, we see that $\mathfrak{o}_\mathfrak{u}$ is a local ring in the sense of Krull.

Assume now that $\mathfrak{o}$ is a semi-local ring[6] and that $\mathfrak{u}$ is one of the maximal prime ideals in $\mathfrak{o}$. Let $\bar{\mathfrak{o}}$ be the completion of $\mathfrak{o}$; there corresponds to $\mathfrak{u}$ an idempotent ϵ in $\bar{\mathfrak{o}}$. We shall prove the following results:

PROPOSITION 5. *The ring* $\mathfrak{o}/\mathfrak{s}$ *is isomorphic with the subring* $\mathfrak{o}\epsilon$ *of the ring* $\bar{\mathfrak{o}}\epsilon$. *This isomorphism may be extended to an isomorphism of the completion of* $\mathfrak{o}_\mathfrak{u}$ *with* $\bar{\mathfrak{o}}\epsilon$.

The first statement will be proved if we show that $\mathfrak{s}$ coincides with the set of elements $x \in \mathfrak{o}$ which satisfy the condition $x\epsilon = 0$. If x is any

[4] Cf. Krull, *Dimensionstheorie in Stellenringen*, J. Reine Angew. Math. vol. 179 (1938) p. 204 or my paper quoted in footnote 1.

[5] This result, together with Proposition 1 above, yields a proof of a theorem of Krull; cf. Krull, *Primidealketten in allgemeinen Ringbereiche*, Sitzungsberichte der Heidelberger Akademie, 1928, p. 7.

[6] A semi-local ring is a Noetherian ring (that is, the maximal chain condition holds in the ring) in which there exist only a finite number of maximal prime ideals. For the proofs of the results on semi-local rings which are used in this paper, cf. my paper quoted in footnote 1.

element of $\mathfrak{o}$, we may write $x = x\epsilon + x(1-\epsilon)$, and we know that $1-\epsilon \in \mathfrak{u}^n\bar{\mathfrak{o}}$ for every n. If $x\epsilon = 0$, we have $x \in \mathfrak{u}^n\bar{\mathfrak{o}} \cap \mathfrak{o} = \mathfrak{u}^n$ for every n; in particular, x belongs to every primary ideal for $\mathfrak{u}$, whence $x \in \mathfrak{s}$. If $x \in \mathfrak{s}$, we have $x \in \mathfrak{u}^n$ for every n ($\mathfrak{u}^n$ is primary because $\mathfrak{u}$ is a *maximal* prime ideal), whence $x\epsilon \in \mathfrak{u}^n\bar{\mathfrak{o}}\epsilon$. Since $\mathfrak{u}\bar{\mathfrak{o}}\epsilon$ is the ideal of non-units in $\bar{\mathfrak{o}}\epsilon$, it follows that $x\epsilon = 0$.

If we identify $\mathfrak{o}/\mathfrak{s}$ with $\mathfrak{o}\epsilon$, every element of S^* is a unit in $\bar{\mathfrak{o}}\epsilon$. In fact, if $y \in S$, we have $y = y\epsilon + y(1-\epsilon)$, $y(1-\epsilon) \in \mathfrak{u}\bar{\mathfrak{o}}$. If we had $y \in \mathfrak{u}\bar{\mathfrak{o}}$, we would have $y \in \mathfrak{u}\bar{\mathfrak{o}} \cap \mathfrak{o} = \mathfrak{u}$, which is not the case. It follows that $\bar{\mathfrak{o}}\epsilon$ contains the ring $\mathfrak{o}_{\mathfrak{u}}$. The ring $\bar{\mathfrak{o}}\epsilon$ is a complete ring with $\mathfrak{u}\bar{\mathfrak{o}}\epsilon$ as unique maximal prime ideal; it is clear that $\mathfrak{o}\epsilon$ (and, a fortiori, $\mathfrak{o}_{\mathfrak{u}}$) is dense in $\bar{\mathfrak{o}}\epsilon$. In order to prove that $\bar{\mathfrak{o}}\epsilon$ is the completion of $\mathfrak{o}_{\mathfrak{u}}$, it is sufficient to prove that $\mathfrak{o}_{\mathfrak{u}}$ is topologically a subspace of $\bar{\mathfrak{o}}\epsilon$. We show first that $\mathfrak{u}^n\bar{\mathfrak{o}}\epsilon \cap \mathfrak{o}\epsilon = \mathfrak{u}^n\mathfrak{o}\epsilon$ for every n. Let $x\epsilon$ ($x \in \mathfrak{o}$) be an element of $\mathfrak{u}^n\bar{\mathfrak{o}}\epsilon \cap \mathfrak{o}\epsilon$; we have $x = x\epsilon + x(1-\epsilon) \in \mathfrak{u}^n\bar{\mathfrak{o}}$, whence $x \in \mathfrak{u}^n\bar{\mathfrak{o}} \cap \mathfrak{o} = \mathfrak{u}^n$, $x\epsilon \in \mathfrak{u}^n\epsilon$. The ideal $\mathfrak{u}^n\bar{\mathfrak{o}} \cap \mathfrak{o}_{\mathfrak{u}}$ is equal to $((\mathfrak{u}^n\bar{\mathfrak{o}} \cap \mathfrak{o}_{\mathfrak{u}}\mathfrak{o}) \cap \epsilon)\mathfrak{o}_{\mathfrak{u}} = (\mathfrak{u}^n\mathfrak{o}\epsilon)\mathfrak{o}_{\mathfrak{u}} = \mathfrak{u}^n\mathfrak{o}_{\mathfrak{u}}$; Proposition 5 is thereby completely proved.

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