CONTINUED FRACTIONS AND BOUNDED ANALYTIC FUNCTIONS

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1. **Introduction.** In this paper we use the characterization given by Schur [6][1] for analytic functions bounded in the unit circle together with the Stieltjes integral representation of F. Riesz [5] for analytic functions with positive real parts, to obtain a new proof of a theorem [8] characterizing totally monotone sequences in terms of Stieltjes continued fractions. In the first place, Schur used an algorithm which he called a "continued fraction-like" algorithm. We begin by constructing from this an *actual* continued fraction algorithm, and we then characterize the class of analytic functions bounded in the unit circle in terms of this continued fraction. Next, we obtain by a simple transformation a continued fraction for functions with positive real parts.[2] This along with the above mentioned-theorem of F. Riesz leads to the theorem [8, pp. 165–166] that the sequence $\{c_p\}$ is totally monotone if and only if the power series $c_0-c_1z+c_2z^2-\cdots$ is the expansion of a continued fraction of the form

$$\frac{g_0}{1}+\frac{g_1z}{1}+\frac{(1-g_1)g_2z}{1}+\frac{(1-g_2)g_3z}{1}+\cdots,$$

where $g_0\geqq 0$, $0\leqq g_p\leqq 1$, $p=1, 2, 3, \cdots$.

2. **An actual continued fraction algorithm derived from the "continued fraction-like" algorithm of Schur.** The continued fraction which we shall consider is as follows:[3]

$$(2.1)\qquad \alpha_0+\frac{(1-\alpha_0\bar{\alpha}_0)z}{\bar{\alpha}_0z}\;\frac{1}{-\alpha_1}+\frac{(1-\alpha_1\bar{\alpha}_1)z}{\bar{\alpha}_1z}\;\frac{1}{-\alpha_2}+\cdots,$$

in which the α_p are complex constants with moduli not exceeding unity, and z is a complex variable. It will be convenient to suppose that if for some p, $|\alpha_p|=1$, then the continued fraction *terminates* with the first identically vanishing partial numerator.

The pth approximant of (2.1) will be denoted by $A_p(z)/B_p(z)$, where $A_0(z)=\alpha_0$, $B_0(z)=1$, $A_1(z)=z$, $B_1(z)=\bar{\alpha}_0z$, and the other nu-

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] A special case of this was given in [9, p. 415].

[3] Hamel [2] used a somewhat different continued fraction for the purpose of characterizing analytic functions bounded in the unit circle. He was obliged to use an unconventional definition of convergence.

merators and denominators are to be computed by means of the recursion formulas

$$\begin{aligned} A_{2p}(z) &= \alpha_p A_{2p-1}(z) - A_{2p-2}(z), \\ B_{2p}(z) &= \alpha_p B_{2p-1}(z) - B_{2p-2}(z), \\ A_{2p+1}(z) &= \bar{\alpha}_p z A_{2p}(z) + (1 - \alpha_p\bar{\alpha}_p) z A_{2p-1}(z), \\ B_{2p+1}(z) &= \bar{\alpha}_p z B_{2p}(z) + (1 - \alpha_p\bar{\alpha}_p) z B_{2p-1}(z), \end{aligned} \qquad p = 1, 2, 3, \cdots. \tag{2.2}$$

If we write

$$\pi_p = \prod_{q=0}^{p} (1 - \alpha_q\bar{\alpha}_q),$$

then we have the "determinant formulas"

$$A_{2p+1}(z)B_{2p}(z) - A_{2p}(z)B_{2p+1}(z) = (-1)^p z^{p+1}\pi_p, \tag{2.3}$$

$$A_{2p+2}(z)B_{2p}(z) - A_{2p}(z)B_{2p+2}(z) = (-1)^p z^{p+1}\pi_p\alpha_{p+1}. \tag{2.4}$$

These may be readily derived from the recursion formulas.

We shall now establish the following theorem:

THEOREM A. *A function $f(z)$ is analytic and has modulus not greater than unity for $|z| < 1$ if and only if it is equal to a terminating continued fraction of the form* (2.1), *or is the limit for $|z| < 1$ of the sequence of even approximants of a nonterminating continued fraction of the form* (2.1).

PROOF. Except in the case where $|\alpha_0| = 1$, so that the continued fraction is equal to the constant α_0, the moduli of the even approximants are all *less* than 1 for $|z| < 1$. In fact, consider the linear fractional transformation

$$t = t(w) = \frac{\alpha - zw}{1 - \bar{\alpha}zw} = \alpha + \frac{(1 - \alpha\bar{\alpha})z}{\bar{\alpha}z - (1/w)}, \qquad |\alpha| < 1,$$

of the w-plane into the t-plane, the transformation depending upon the parameter z. If $|z| < 1$, this transformation has the property that $|t| < 1$ for $|w| \leqq 1$. The same is true of the product of two or more such transformations, and inasmuch as $A_{2p}(z)/B_{2p}(z)$ is equal to the product of p such transformations applied to the point $w = \alpha_p$, we conclude that

$$\left| \frac{A_{2p}(z)}{B_{2p}(z)} \right| < 1 \quad \text{for} \quad |z| < 1. \tag{2.5}$$

In the case where (2.1) does not terminate, the sequence of even approximants converges uniformly for $|z| \leqq r$ for every positive constant r less than 1, and represents an analytic function with modulus not greater than 1 for $|z| < 1$. In fact, from the determinant formula (2.4) we have the relation

$$\frac{A_{2p+2}(z)}{B_{2p+2}(z)} - \frac{A_{2p}(z)}{B_{2p}(z)} = \frac{(-1)^p \pi_p \alpha_{p+1} z^{p+1}}{B_{2p+2}(z) B_{2p}(z)}. \tag{2.6}$$

It is easy to see from the recursion formulas and (2.3), (2.4) that the denominators $B_{2p}(z)$ are different from zero for $|z| \leqq 1$. Hence, the expansion in ascending powers of z of the right member of (2.6) begins with the $(p+1)$th power, or a higher power, of z. Consequently, there exists a power series $P(z) = c_0 - c_1 z + c_2 z^2 - \cdots$ which agrees term by term with the series for $A_{2p}(z)/B_{2p}(z)$ for more and more terms as p is increased. Inasmuch as the coefficients in the expansion of this rational function do not exceed 1 in numerical value by virtue of (2.5), we conclude that $|c_p| \leqq 1$, $p = 0, 1, 2, \cdots$, so that $P(z)$ converges for $|z| < 1$. Moreover, if we put $A_{2p}(z)/B_{2p}(z) = a_0 - a_1 z + a_2 z^2 - \cdots$, then if $|z| \leqq r < 1$,

$$\left| P(z) - \frac{A_{2p}(z)}{B_{2p}(z)} \right| = \left| \sum_{q=0}^{\infty} (-1)^q (c_q - a_q) z^q \right| \leqq \frac{2r^{p+1}}{1-r}, \tag{2.7}$$

from which we conclude that the sequence $\{A_{2p}(z)/B_{2p}(z)\}$ converges uniformly to $P(z)$ for $|z| \leqq r < 1$; and from (2.5) it follows that $|P(z)| \leqq 1$ for $|z| < 1$.

We must now show, conversely, that if $P(z) = c_0 - c_1 z + \cdots$ is any function which is analytic and has modulus not greater than unity for $|z| < 1$, then there exists a continued fraction of the form (2.1) such that (2.7) holds. To do this, we define the function $P_1(z)$ by

$$P_1(z) = \frac{1}{z} \frac{c_0 - P(z)}{1 - \bar{c}_0 P(z)} = \frac{c_1 - c_2 z + c_3 z^2 - \cdots}{1 - \bar{c}_0 (c_0 - c_1 z + c_2 z^2 - \cdots)}.$$

It is clear that $|c_0| \leqq 1$, being the value of $P(0)$, and that if $|c_0| = 1$, then $P(z) \equiv c_0$. In either case, we put $\alpha_0 = c_0$. If we suppose that $|c_0| < 1$, then, from the character of the above transformation and from Schwarz's lemma, it follows that $P_1(z)$ is analytic and has modulus not exceeding unity for $|z| < 1$. Take $\alpha_1 = P_1(0)$. If $|\alpha_1| = 1$, then $P_1(z) \equiv \alpha_1$. If, however, $|\alpha_1| < 1$, we write

$$P_2(z) = \frac{1}{z} \frac{\alpha_1 - P_1(z)}{1 - \bar{\alpha}_1 P_1(z)}, \qquad \alpha_2 = P_2(0).$$

The function $P_2(z)$ is analytic and has modulus not greater than unity for $|z|<1$. If $|\alpha_2|=1$, then $P_2(z)\equiv\alpha_2$, while if $|\alpha_2|<1$, the process may be continued. In this way we obtain a finite or infinite sequence of functions

$$P_0(z) = P(z),\ P_1(z),\ P_2(z),\ \cdots, \tag{2.8}$$

satisfying the relations

$$P_{k+1}(z) = \frac{1}{z}\,\frac{\alpha_k - P_k}{1-\bar{\alpha}_k P_k}, \qquad P_k = \frac{\alpha_k - zP_{k+1}}{1-\bar{\alpha}_k z P_{k+1}}, \qquad \alpha_k = P_k(0). \tag{2.9}$$

From these we derive immediately a formal terminating or nonterminating continued fraction expansion (2.1) for $P(z)$. In the terminating case, the expansion is obviously valid, being in fact an identity. In the nonterminating case we have, for arbitrary k, the identity

$$P(z) = \frac{P_k(z)A_{2k-1}(z) - A_{2k-2}(z)}{P_k(z)B_{2k-1}(z) - B_{2k-2}(z)},$$

so that, by (2.3),

$$P(z) - \frac{A_{2k-2}(z)}{B_{2k-2}(z)} = \frac{P_k(z)(-1)^{k-1}\pi_{k-1}z^k}{B_{2k}(z)\left[P_k(z)B_{2k-1}(z) - B_{2k-2}(z)\right]}.$$

From this it readily follows that the power series for $A_{2k-2}(z)/B_{2k-2}(z)$ agrees term by term with $P(z)$ for more and more terms as k is increased, and consequently (2.7) holds.

This completes the proof of Theorem A. This proof is the same as that of Schur [6], except that we have used the formulas and notation of continued fractions. It should be observed that the constants α_p are uniquely determined by means of the given function $P(z)$, either as an infinite sequence in the one case or as a finite sequence in the other.

3. **A continued fraction expansion for functions with positive real parts.** A function $k(z)$ is analytic and has a nonnegative real part for $|z|<1$ and has the value 1 for $z=0$ if and only if there exists a function $P(z)$ which is analytic and has modulus not greater than 1 for $|z|<1$, such that

$$k(z) = \frac{1+zP(z)}{1-zP(z)}. \tag{3.1}$$

This follows from Schwarz's lemma and the relation

$$R(k(z)) = \frac{1 - |zP(z)|^2}{|1 - zP(z)|^2}.$$

We shall now obtain a continued fraction expansion for $k(z)$.

Let $\beta_0 = 1$, and determine $\beta_1, \beta_2, \cdots$ by the relations

$$\beta_{p+1} = -\frac{\bar{\alpha}_p - \beta_p}{1 - \alpha_p \beta_p}, \qquad p = 0, 1, 2, \cdots, \tag{3.2}$$

where the α_p are the numbers appearing in the continued fraction (2.1) for the function $P(z)$ determined by (3.1). The numbers β_p have moduli equal to 1, and form an infinite sequence or a finite sequence according as the sequence $\{\alpha_p\}$ is infinite or finite, respectively. Instead of the functions (2.9) we now introduce functions $h_p(z)$ by

$$h_p(z) = \frac{1 - \beta_p P_p(z)}{1 + z\beta_p P_p(z)}, \qquad p = 0, 1, 2, \cdots. \tag{3.3}$$

By means of (2.9) and (3.2) we then find that

$$h_p(z) = \frac{\beta_p - \bar{\alpha}_p}{\beta_{p+1} - \beta_p z + (\beta_{p+1} + \bar{\alpha}_p) z h_{p+1}(z)}, \quad p = 0, 1, 2, \cdots. \tag{3.4}$$

Remembering that $\beta_0 = 1$, we therefore have the formal continued fraction expansion:

$$\frac{1 - P(z)}{1 + zP(z)} = \frac{\beta_0 - \bar{\alpha}_0}{\beta_1 - \beta_0 z} + \frac{(\beta_1 + \bar{\alpha}_0)(\beta_1 - \bar{\alpha}_1)z}{\beta_2 - \beta_1 z} + \frac{(\beta_2 + \bar{\alpha}_1)(\beta_2 - \bar{\alpha}_2)z}{\beta_3 - \beta_2 z} + \cdots. \tag{3.5}$$

On multiplying both members of (3.5) by $2z/(1-z)$, adding 1 to both members, and then taking reciprocals, we have, using (3.1),

$$k(z) = \frac{1+z}{1-z} + \frac{2(\beta_0 - \bar{\alpha}_0)z}{\beta_1 - \beta_0 z} + \frac{(\beta_1 + \bar{\alpha}_0)(\beta_1 - \bar{\alpha}_1)z}{\beta_2 - \beta_1 z} + \frac{(\beta_2 + \bar{\alpha}_1)(\beta_2 - \bar{\alpha}_2)z}{\beta_3 - \beta_2 z} + \cdots. \tag{3.6}$$

In case $|\alpha_p| < 1$, $p = 0, 1, 2, \cdots, n-1$, $|\alpha_n| = 1$, this continued fraction terminates, the last partial quotient being equal to

$$\frac{(\beta_n + \bar{\alpha}_{n-1})(1 - \alpha_n \beta_n)z}{1 + \alpha_n \beta_n z};$$

while if $|\alpha_p|<1$, $p=0, 1, 2, \cdots$, the continued fraction does not terminate. In the first case, $k(z)$ is of course equal to the continued fraction. In the second case, the continued fraction converges uniformly in the neighborhood of the origin by a well known theorem [4, p. 259]. An easy argument (cf. [9, pp. 415–416]) then shows that its value is $k(z)$.

If, in particular, $k(z)$ is real when z is real, then $\beta_p=1$, $p=0, 1, 2, \cdots$, the α_p are real, and the continued fraction for $k(z)$ can be thrown by means of an equivalence transformation into the form

$$(3.7)\qquad k(z)=\frac{1+z}{1-z}\cdot\frac{1}{1}\,\frac{}{+}\,\frac{g_1w}{1}\,\frac{}{+}\,\frac{(1-g_1)g_2w}{1}\,\frac{}{+}\,\frac{(1-g_2)g_3w}{1}\,\frac{}{+}\cdots,$$

where $w=4z/(1-z)^2$ and $g_p=(1-\alpha_{p-1})/2$, $p=1, 2, 3, \cdots$. Thus $0\leqq g_p\leqq 1$, the continued fraction terminating in case equality holds for some value of p.

It is readily seen that, conversely, if the α_p are given, then the function $k(z)$ given by (3.6) or (3.7) has the stated properties. We therefore have the following theorem:

THEOREM B. *A function $k(z)$ is analytic and has a nonnegative real part for $|z|<1$, and is equal to* 1 *for $z=0$, if and only if it has a continued fraction expansion of the form* (3.6), *where the α_p are constants with moduli not greater than* 1, *and the β_p are given in terms of the α_p by* (3.2), *β_0 being equal to* 1. *If $k(z)$ is real when z is real, the continued fraction can be thrown into the form* (3.7).

4. A characterization of totally monotone sequences in terms of continued fractions.

A sequence $\{c_p\}$ of real numbers is called *totally monotone* if all the differences $\Delta^m c_n=c_n-C_{m,1}c_{n+1}+C_{m,2}c_{n+2}-\cdots+(-1)^m C_{m,m}c_{n+m}$, $m, n=0, 1, 2, \cdots$, are nonnegative. If $c_0=0$, then $c_p=0$ for $p=1, 2, 3, \cdots$. Excepting in this trivial case, we may normalize by dividing every member of the sequence by c_0, and may thus assume that $c_0=1$.

Hausdorff [3] showed that $\{c_p\}$ is totally monotone if and only if there exists a bounded nondecreasing function $\phi(u)$ such that

$$(4.1)\qquad c_p=\int_0^1 u^p d\phi(u),\qquad p=0, 1, 2, \cdots.$$

This is equivalent to saying that

$$(4.2)\qquad c_0-c_1w+c_2w^2-\cdots=\int_0^1\frac{d\phi(u)}{1+wu}.$$

We now turn our attention to a theorem of F. Riesz [5] which, along with Theorem B, will furnish a characterization of totally monotone sequences in terms of continued fractions.

THEOREM C. *A function $k(z)$ is analytic and has a nonnegative real part for $|z|<1$, and is equal to* 1 *for $z=0$, if and only if it has an integral representation of the form*

$$k(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\alpha(t), \tag{4.3}$$

where $\alpha(t)$ is a nondecreasing function such that $\alpha(0)=0$, $\alpha(2\pi)=1$. The function $\alpha(t)$ is determined uniquely to an additive constant at all its points of continuity by $k(z)$.

PROOF. Put

$$\begin{aligned} k(z) &= 1 + a_1 z + a_2 z^2 + \cdots \\ &= 1 + \sum_{p=1}^{\infty} (b_p + ic_p) r^p (\cos p\theta + i \sin p\theta), \end{aligned} \tag{4.4}$$

where $z=re^{i\theta}$, $0<r<1$. Then, if

$$u(r, \theta) = \Re(k(z)) = 1 + \sum_{p=1}^{\infty} r^p (b_p \cos p\theta - c_p \sin p\theta) \geqq 0,$$

we conclude that the function

$$\alpha_r(t) = \frac{1}{2\pi}\int_0^t u(r, \theta) d\theta, \qquad 0 \leqq t \leqq 2\pi,$$

is a nondecreasing function of t; and $\alpha_r(0)=0$, $\alpha_r(2\pi)=1$. Moreover,

$$\int_0^{2\pi} d\alpha_r(t) = 1, \qquad \int_0^{2\pi} 2\cos qt d\alpha_r(t) = r^q b_q,$$
$$\int_0^{2\pi} -2\sin qt d\alpha_r(t) = r^q c_q.$$

Using a well known theorem, we may now determine a nondecreasing function $\alpha(t)$ such that $\alpha(0)=0$, $\alpha(2\pi)=1$, and a sequence of values of r approaching 1, such that these equations go over into

$$\int_0^{2\pi} d\alpha(t) = 1, \qquad \int_0^{2\pi} 2\cos qt d\alpha(t) = b_q,$$
$$\int_0^{2\pi} -2\sin qt d\alpha(t) = c_q.$$

When these values are substituted in (4.4), that series may be written as the integral of a geometric series. On summing the latter, we obtain (4.3). The essential uniqueness of the function $\alpha(t)$ follows from the fact that an arbitrary continuous function can be approximated uniformly by a trigonometric polynomial [**5**, pp. 38–39].

Conversely, if $\alpha(t)$ is any bounded nondecreasing function such that $\alpha(0)=0$, $\alpha(2\pi)=1$, then one may verify at once that the function $k(z)$ defined by (4.3) has the required properties.

We now make in (4.3) the change of variable $u=\sin^2(t/2)$, and that formula becomes

$$(4.5)\quad k(z)=\frac{1+z}{1-z}\int_0^1\frac{d\phi(u)}{1+wu}-iw\int_0^1\frac{[u(1-u)]^{1/2}\cdot d\psi(u)}{1+wu},$$

where $\alpha(t)+\alpha(2\pi-t)=\psi(u)$, $1-\alpha(2\pi-t)=\phi(u)$, $w=4z/(1-z)^2$. The function $\psi(u)$ is of bounded variation, and $\phi(u)$ is bounded and nondecreasing: $\phi(0)=0$, $\phi(1)=1$. The second integral in (4.5) vanishes identically if and only if $k(z)$ is real when z is real. On the other hand, $k(z)$ is of this character if and only if it has an expansion of the form (3.7). On putting these two facts together we obtain the relation

$$(4.6)\quad \int_0^1\frac{d\phi(u)}{1+wu}=\frac{1}{1}+\frac{g_1w}{1}+\frac{(1-g_1)g_2w}{1}+\frac{(1-g_2)g_3w}{1}+\cdots.$$

Recalling now the definition of a totally monotone sequence, and the statement made concerning (4.2), we have the following theorem:

THEOREM D. *The sequence $\{c_p\}$ of real numbers, of which $c_0=1$, is totally monotone if and only if the power series $c_0-c_1w+c_2w^2-\cdots$ has a continued fraction expansion of the form*

$$(4.7)\quad \frac{1}{1}+\frac{g_1w}{1}+\frac{(1-g_1)g_2w}{1}+\frac{(1-g_2)g_3w}{1}+\cdots,$$

where $0\leqq g_p\leqq 1$, $p=1, 2, 3, \cdots$, it being agreed that the continued fraction shall terminate in case some partial numerator vanishes identically.

An interesting consequence of this theorem is the fact that there exists a sequence of polynomials $G_1(g_1)$, $G_2(g_1, g_2)$, $G_3(g_1, g_2, g_3)$, $\cdots$, where G_p depends upon p variables, such that every totally monotone sequence with $c_0=1$ can be represented parametrically in the form

$$c_p=G_p(g_1, g_2, \cdots, g_p),\qquad p=0, 1, 2, \cdots\ (G_0=1),$$

where $0\leqq g_p\leqq 1$, $p=1, 2, 3, \cdots$. Conversely, every sequence of this form is totally monotone. To obtain the polynomials G_p it is but necessary to observe that the power series expansion of (4.7) is the series $c_0 - c_1 w + c_2 w^2 + \cdots$ where the c_p are polynomials in the g_p. These may be most conveniently calculated by means of formulas given by Stieltjes [7, pp. 419–420]. We mention only that

$$(4.8) \qquad G_1(g_1) = g_1, \qquad G_2(g_1, g_2) = g_1^2(1 - g_2) + g_1 g_2.$$

F. Riesz [5, p. 56] showed that a sequence $\{c_p\}$, where $c_0 = 1$, is totally monotone if and only if for every n the point $(c_0, c_1, \cdots, c_{n-1})$ lies in the convex extension of the continuous curve given in parametric form by $(1, u, u^2, \cdots, u^{n-1})$, $0 \leqq u \leqq 1$. The polynomials G_p therefore furnish a parametric representation for the points of this convex set. This can be verified for $n = 3$ by means of (4.8).

5. **Continued fraction transformations.** If the α_p are real in (3.5), that relation may be written in the form

$$(5.1) \qquad \frac{(1-z)}{2}\,\frac{1 - P(z)}{1 + zP(z)} = \frac{g_1}{1} + \frac{(1 - g_1)g_2 w}{1} + \frac{(1 - g_2)g_3 w}{1} + \cdots,$$

where $g_p = (1 - \alpha_{p-1})/2$, $p = 1, 2, 3, \cdots$, and $w = 4z/(1-z)^2$. To indicate the dependence of $P(z)$ upon the α_p we shall now write $P(z) = (z; \alpha_0, \alpha_1, \alpha_2, \cdots)$. The following relations may be readily verified [6]:

$$(5.2) \qquad -P(z) = (z; -\alpha_0, -\alpha_1, -\alpha_2, \cdots),$$

$$(5.3) \qquad P(-z) = (z; \alpha_0, -\alpha_1, \alpha_2, -\alpha_3, \cdots),$$

$$(5.4) \qquad P(z^n) = (z; \alpha_0, 0, \cdots, 0, \alpha_1, 0, \cdots, 0, \alpha_2, 0, \cdots),$$

where 0 occurs $n-1$ times between α_p and α_{p+1}. We observe that the effect of replacing α_{p-1} by $-\alpha_{p-1}$ is to replace g_p by $1 - g_p$; and that the effect of replacing α_{p-1} by 0 is to replace g_p by 1/2. These facts together with the preceding relations enable us to obtain a number of transformations of the continued fraction in (5.1).

Let us denote the right member of (5.1) by $F(w)$, and write $F(w) = [w; g_1, g_2, g_3, \cdots]$. On replacing $P(z)$ by $-P(z)$ and α_{p-1} by $-\alpha_{p-1}$, that is, g_p by $1 - g_p$, $p = 1, 2, 3, \cdots$, we obtain at once the relation [8, p. 166; 9, p. 416]

$$(5.5) \qquad \frac{1 - F(w)}{1 + wF(w)} = [w; 1 - g_1, 1 - g_2, 1 - g_3, \cdots].$$

Similarly, on replacing $P(z)$ by $P(-z)$ and α_p by $(-1)^p \alpha_p$ we get [1, p. 191]

$$(5.6)\qquad 1 - F(-w/(1+w)) = [w; 1 - g_1, g_2, 1 - g_3, g_4, \cdots].$$

Using (5.4), replacing $f(z)$ by $f(z^n)$ in (5.1), we obtain a relation of the form

$$(5.7)\qquad \frac{F(U_n(w))}{V_n(w) + W_n(w)F(U_n(w))} = [w; g_1, 1/2, \cdots, 1/2, g_2, 1/2, \cdots, 1/2, g_3, 1/2, \cdots],$$

where 1/2 appears $n-1$ times in each place, and where $U_n(w)$, $V_n(w)$, $W_n(w)$ are rational functions of w given by

$$U_n(w) = \frac{4w^n}{\{((1+w)^{1/2}+1)^n - ((1+w)^{1/2}-1)^n\}^2},$$

$$V_n(w) = (1+w)^{1/2}\frac{\{((1+w)^{1/2}+1)^n - ((1+w)^{1/2}-1)^n\}^2}{((1+w)^{1/2}+1)^{2n} - ((1+w)^{1/2}-1)^{2n}},$$

$$W_n(w) = \frac{w\{((1+w)^{1/2}+1)^{n-1} - ((1+w)^{1/2}-1)^{n-1}\}^2}{2w^{n-1} - \{((1+w)^{1/2}+1)^{2n-1} - ((1+w)^{1/2}-1)^{2n-1}\}}.$$

Here, that branch of $(1+w)^{1/2}$ is to be taken which reduces to 1 for $w=0$.

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