

function depending on n arbitrary functions can be obtained. As mentioned above, Chapter VII also contains separating variables and expansion of boundary conditions in Fourier series. The equations treated include the vibrating string, the cable, fluid flow, and heat flow. These equations have already been derived from physics in Chapter III.

Chapter VIII on nonlinear equations of second order is devoted largely to obtaining solutions of Monge's equation depending on arbitrary functions.

The book contains very few examples from physics except in the boundary value problems solved with Fourier series. However, the book contains an exceptionally large number of problems in which the student is asked to find a solution depending on arbitrary constants or functions. In a large proportion of these the partial differential equation is proposed as a problem in geometry.

The multiple integration approach to partial differential equations is not touched upon. The concept of characteristics of a second order equation and the classification of second order equations does not appear. Also there is no mention of successive integrations.

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The calculi of lambda-conversion. By Alonzo Church. (Annals of Mathematics Studies, no. 6.) Princeton, Princeton University Press; London, Humphrey Milford and Oxford University Press, 1941. 2+77 pp. \$1.25.

This is a brief and attractively written introduction to the remarkable formal systems discovered by Church and called by him calculi of lambda-conversion. These systems were developed by Church in collaboration with his students, S. C. Kleene and J. B. Rosser. The present booklet, which is lithographed, is in most respects a considerable improvement over the same author's mimeographed Princeton lecture notes of 1936, of which it may be considered a revision. The notation has been simplified and improved, and the treatment of Gödel numbers is much simpler than in the former version. The proof of the fundamental Church-Rosser consistency theorem is now given in full detail, and the section on recursive arithmetic has been considerably expanded.

Nevertheless the text of the present version totals only 71 pages. This brevity is partly accounted for by the plan which the author has wisely adopted of considering only the calculus of lambda-conversion proper in full detail. In fact, the first four chapters are devoted to this, the most elementary of the lambda-calculi. The more com-

plicated, but also more useful, systems called the calculus of λ - K -conversion, the calculus of restricted λ - K -conversion, and the calculus of λ - δ -conversion, as well as the applications of the latter to symbolic logic, are discussed only very briefly in the fifth and final chapter.

The first chapter is introductory, and explains the intended interpretation of the symbols by means of examples. The second chapter presents the lambda-calculus as a formal system; and a very simple formal system it is. There are three kinds of symbols (the symbol λ , parentheses, and letters for variables), one undefined operation called *application* and denoted by juxtaposition, and three rules of procedure called the rules of lambda-conversion. There are no axioms and no theorems, the subject being developed metatheoretically. (This is a matter of convenience; it would be possible to introduce axioms in terms of a symbol for interconvertibility of formulas, if that were considered desirable.) Chapter II also deals with the important notion of *normal form*, and contains the proof of the Church-Rosser consistency theorem. The author intends only those formulas which have a normal form to be meaningful, although there exist quite simple formulas with no normal form and hence with no interpretation. It has been shown that no effective procedure exists for determining in all cases whether a given formula has a normal form. This fact is related to the incompleteness property which, according to the results of Gödel, any system which is consistent and adequate for formal logic must have.

In the third chapter the positive integers are defined in terms of the lambda symbolism. This is possible because the system contains formulas which, applied to a function, have the effect of iterating it a given number of times. The familiar operations of addition, multiplication, and exponentiation of integers are also defined. It is shown that an integral-valued function of positive integers is general recursive if and only if it is lambda-definable, that is, definable in terms of the lambda symbolism. Reasons are given for identifying the notion of an effectively calculable function of positive integers with that of a lambda-definable function, or equivalently of a general recursive function.

Chapter four deals with Gödel numbers. It is convenient, in formal systems which contain a notation for the integers, to develop a method of numbering the formulas of the system which is definable in terms of the system itself. This enables the system to talk about itself, roughly speaking, and sometimes, indeed, even to contradict itself (but that cannot happen here). Since it is simpler to assign the same Gödel number to two formulas if they differ only in the letters

used to represent bound variables, the author first assigns to every well-formed formula a special kind of formula called a *combination*, and then enumerates the combinations. Formulas which differ only in the symbols for bound variables are assigned the same combination. In connection with combinations, the close connection between the lambda-calculi and the combinatory logic of M. Schönfinkel and H. B. Curry is brought out.

The restricted calculus of lambda-conversion which is treated in the first four chapters is not an adequate basis for logic or mathematics. In particular, constant functions cannot be defined because of the rule that λxM (which is interpreted to mean the function that M is of x) is not well-formed unless x occurs as a free variable in M . The fifth chapter deals briefly with modified systems which avoid the objection mentioned (the lack of constant functions), but which give rise to new difficulties, as the author points out. Of these, the calculi of λ - K -conversion and of restricted λ - K -conversion arise by changing the definition of well-formed formula to allow more well-formed formulas. The λ - δ -calculus involves the introduction of a new symbol δ which behaves somewhat like a sign of equality.

Finally the author outlines his method of defining within the λ - δ -calculus a system of symbolic logic. This is accomplished by identifying the truth values *truth* and *falsity* with the formulas for the integers 2 and 1 respectively. The provable formulas of the logic are then those formulas of the calculus which are convertible to the formula for the integer 2, that is to the formula for truth, according to the rules of λ - δ -conversion. Operations of negation, conjunction, and disjunction are defined, as well as an existential quantifier and a selection operator. All these have properties differing from those of the corresponding operations of classical logic, and resembling somewhat those of the intuitionist logic. It is indicated that there are difficulties involved in defining a universal quantifier. Although this logic seems peculiar in some ways, it has the advantage over certain other systems that there exists for it an elementary consistency proof, namely the analogue for the λ - δ -calculus of the Church-Rosser theorem. The further elaboration of this system of logic is left for another book.

In this book Church has done a first-rate job of exposition, which should be examined by everyone interested in the foundations of mathematics. Not that it is all easy reading; that would be impossible in the nature of the subject. The fundamental notions of the lambda-calculus have what seems to be a simple interpretation, but this interpretation does not seem to help one to follow the details. Perhaps this is partly because of the long arrays of symbols required to

define most of the important formulas. Verification that the formulas have the desired properties must proceed formally according to the conversion rules, and the work is often tedious.

Two interesting points concerning these calculi may be noted. One is that arithmetic is developed first, and logical notions are then defined in terms of arithmetic, whereas in most such formal systems, like the Whitehead-Russell system, logical notions come first and arithmetic is defined in terms of logic. The other point is that these calculi seem to support the views of the effectivists (including the intuitionists) who would exclude from mathematics the axiom of choice and in general those concepts which are not constructively definable. In particular the prominent role played here by the integers is suggestive of the viewpoint of Kronecker. However, it should be said that although the lambda-calculi make it possible to state more exactly what is meant by an effective definition, and to arrive at interesting results concerning the concept of effectiveness, one can use the methods of this book without committing oneself to the effectivist position. Presumably it is quite possible to erect a non-effectivist mathematics on the basis of these systems.

Church has suggested elsewhere that his lambda notation might be useful in avoiding the ambiguity which results (especially in dealing with function spaces) from the bad habit of using the notation $f(x)$ to stand sometimes for a function as a single entity, and sometimes for the value of the function when the argument is x . It is interesting to note that Whitehead and Russell use the notation $f(\hat{x})$ for the first of these concepts in place of Church's $\lambda x \cdot f(x)$ or $\lambda x(fx)$, and that our elementary textbooks at least provide for the distinction by the use of different letters for the argument; thus $f(x)$ stands for a function, while $f(a)$ or $f(x_0)$ stands for a functional value. Unfortunately the distinction is not always made. It should of course be said that Church's lambda notation is greatly superior in some ways to these other notations, and allows the construction of formulas not representable in them.

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