

# A RECURRENCE FORMULA FOR THE SOLUTIONS OF CERTAIN LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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**1. Introduction.** In a number of recent papers, Bergman<sup>1</sup> has developed the theory of operational methods for transforming analytic functions of a complex variable into solutions of the linear partial differential equation

$$(1.1) \quad L(U) = U_{z\bar{z}} + a(z, \bar{z})U_z + b(z, \bar{z})U_{\bar{z}} + c(z, \bar{z})U = 0,$$

where  $z = x + iy$ ,  $\bar{z} = x - iy$ ,

$$U_z = \frac{1}{2} \left( \frac{\partial U}{\partial x} - i \frac{\partial U}{\partial y} \right), \quad U_{\bar{z}} = \frac{1}{2} \left( \frac{\partial U}{\partial x} + i \frac{\partial U}{\partial y} \right),$$

$$U_{z\bar{z}} = \frac{1}{4} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) = \frac{\Delta U}{4},$$

and where the coefficients  $a(z, \bar{z})$ ,  $b(z, \bar{z})$  and  $c(z, \bar{z})$  are analytic functions of both variables  $z$  and  $\bar{z}$ . The equation (1.1) is equivalent to the system of two real equations

$$\Delta U^{(1)} + 2AU_x^{(1)} + 2BU_y^{(1)} + 2CU_x^{(2)} + 2DU_y^{(2)} + 4c_1U^{(1)} - 4c_2U^{(2)} = 0,$$

$$\Delta U^{(2)} - 2CU_x^{(1)} - 2DU_y^{(1)} + 2AU_x^{(2)} + 2BU_y^{(2)} + 4c_2U^{(1)} + 4c_1U^{(2)} = 0,$$

where

$$U = U^{(1)} + iU^{(2)}; \quad 2A = (a + \bar{a}) + (b + \bar{b}); \quad 2B = i[(\bar{a} - a) - (\bar{b} - b)]; \\ c = c_1 + ic_2; \quad 2D = (a + \bar{a}) - (b + \bar{b}); \quad 2C = i[(a - \bar{a}) + (b - \bar{b})].$$

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<sup>1</sup> S. Bergman, (a) *Zur Theorie der Funktionen, die eine lineare partielle Differentialgleichung befriedigen*, Rec. Math. (Mat. Sbornik) N.S. vol. 44 (1937) pp. 1169-1198; (b) *The approximation of functions satisfying a linear partial differential equation*, Duke Math. J. vol. 6 (1940) pp. 537-561; (c) *Linear operators in the theory of partial differential equations*, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 130-155; (d) *On the solutions of partial differential equations of the fourth order*, to appear later.

Furthermore, if  $a = \bar{b}$  and  $c$  is real, then  $C = D = c_2 = 0$ , and the two differential equations become real and identical.

For equations (1.1), Bergman proved the existence of two functions  $E_1(z, \bar{z}, t)$  and  $E_2(z, \bar{z}, t)$ , called by him "generating functions of the first kind,"<sup>2</sup> with the following properties:

(1) They have the forms

$$E_1(z, \bar{z}, t) = \exp \left( - \int_0^z a(z, \bar{z}) d\bar{z} \right) [1 + z\bar{z}tE_1^*(z, \bar{z}, t)],$$

$$E_2(z, \bar{z}, t) = \exp \left( - \int_0^z b(z, \bar{z}) dz \right) [1 + z\bar{z}tE_2^*(z, \bar{z}, t)],$$

where each  $E_k^*(z, \bar{z}, t)$  has continuous first partial derivatives in  $z, \bar{z}$  and  $t$  for  $|t| \leq 1$  and for  $z$  and  $\bar{z}$  within a certain four-dimensional region.

(2) The classes  $C(E_1)$  and  $C(E_2)$  of functions  $U_1(z, \bar{z})$  and  $U_2(z, \bar{z})$  defined by the formulas

$$(1.2) \quad U_1(z, \bar{z}) = \int_{-1}^1 E_1(z, \bar{z}, t) f(z(1 - t^2)/2) dt / (1 - t^2)^{1/2},$$

$$(1.3) \quad U_2(z, \bar{z}) = \int_{-1}^1 E_2(z, \bar{z}, t) g(\bar{z}(1 - t^2)/2) dt / (1 - t^2)^{1/2},$$

where  $f(\zeta)$  and  $g(\zeta)$  are arbitrary analytic functions of  $\zeta$ , form subsets of solutions of (1.1).

(3) Every solution  $U(z, \bar{z})$  of (1.1) may be written in the form

$$U(z, \bar{z}) = U_1(z, \bar{z}) + U_2(z, \bar{z}),$$

with  $f(\zeta)$  and  $g(\zeta)$  suitably chosen analytic functions.

As was proved by Bergman, to many theorems about analytic functions of a complex variable correspond analogous theorems about functions belonging to classes  $C(E)$  generated by functions  $E$  of the first kind. In particular, if we define as "basic solutions" those corresponding to  $f(z) = z^p$ , that is

$$(1.4) \quad u_p(z, \bar{z}) = \int_{-1}^1 E(z, \bar{z}, t) [z(1 - t^2)/2]^p dt / (1 - t^2)^{-1/2},$$

then every function  $U$  of class  $C(E)$  which is regular in  $|z| \leq r$  may

<sup>2</sup> Generating functions which are considered as not of the first kind are those failing to satisfy property (1). When  $E$  is a generating function not of the first kind, the integration in (1.2) and (1.3) must be taken along a rectifiable curve joining the points  $t = \pm 1$ , but not passing through  $t = 0$ .

be expanded in a series  $U = \sum \alpha_p u_p$  which is uniformly and absolutely convergent in  $|z| \leq r$ .

For example, in the case of the equation

$$(1.5) \quad \Delta U + U = 0,$$

$E(z, \bar{z}, t) = e^{itz}$ , where  $z = re^{i\theta}$  and thus  $r = (z\bar{z})^{1/2}$ . Because of the well known formula for the Bessel function of the first kind

$$J_p(r) = (2/\pi)(1/\Gamma(p + 1/2)) \int_{-1}^1 e^{rit}(r/2)^p (1 - t^2)^{p-1/2} dt,$$

the basic solutions are

$$(1.6) \quad u_p(r, \theta) = (\pi^{1/2}/2)\Gamma(p + 1/2)e^{p\theta i}J_p(r).$$

But for (1.5), successive terms in the expansion  $U = \sum \alpha_p u_p$  can be computed from earlier terms by the use of some recurrence relation satisfied by the Bessel's functions, as for example the relation

$$(1.7) \quad J_p'(r) = (p/r)J_p(r) - J_{p+1}(r).$$

It would likewise be of practical value in the case of other differential equations  $L(U) = 0$  to determine what recurrence relations, if any, are satisfied by the basic solutions  $u_p(r, \theta)$ .

In the present note, recurrence formulas connecting the basic solutions  $u_p(r, \theta)$  are found in the case of differential equations  $L(U) = 0$  for which at least one of the corresponding "generating functions"  $E(z, \bar{z}, t)$  is of the form  $E(z, \bar{z}, t) = \exp f(r, \theta, t)$  where  $f(r, \theta, t)$  is a polynomial in  $t$  containing either only even powers of  $t$  or only odd powers of  $t$ . Obviously, the equation (1.5) is an example of such an equation. Other examples can be found by requiring the coefficients  $a$ ,  $b$  and  $c$  in the equation  $L(U) = 0$  to satisfy certain differential relations.<sup>3</sup>

Our first main result may be stated as follows:

**THEOREM 1.** *Let  $L(U) = 0$  be a partial differential equation of the type (1.1) for which there exists a generating function having one of the forms:*

$$(I) \quad E(z, \bar{z}, t) = \exp P(r, \theta, t),$$

$$(II) \quad E(z, \bar{z}, t) = \exp tP(r, \theta, t),$$

where  $P(r, \theta, t) = a_0(r, \theta) + a_1(r, \theta)t^2 + \dots + a_n(r, \theta)t^{2n}$ , and where the coefficients  $a_j(r, \theta)$  are of class  $C'$  in  $r$  and  $\theta$ . Let  $u_p(r, \theta)$  be the corresponding "basic solutions" of equation  $L(U) = 0$  and let

<sup>3</sup> See reference in footnote 1a, pp. 1194–1195, and also p. 158 of the following article: K. L. Nielsen and B. P. Ramsay, *On particular solutions of linear partial differential equations*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 156–162.

$$(1.8) \quad \alpha_q(r, \theta) = (-1)^q \left[ \sum_{i=q}^n C_{i,q} \frac{\partial a_i}{\partial r} \right] \left( \frac{2}{re^{i\theta}} \right)^q,$$

$$(1.9) \quad \beta_q(r, \theta) = (-1)^q \left[ \sum_{i=q}^n (2j + 1) C_{i,q} a_i \right] \left( \frac{2}{re^{i\theta}} \right)^q.$$

Then, if  $E$  has form (I),

$$(1.10) \quad \frac{\partial u_p}{\partial r} = \frac{p}{r} u_p + \sum_{i=0}^n \alpha_i u_{p+i};$$

whereas, if  $E$  has form (II),

$$(1.11) \quad \frac{\partial u_p}{\partial r} = \frac{p}{r} u_p + \frac{2}{re^{i\theta}} \sum_{i,k=0}^n \frac{\alpha_i \beta_k u_{i+k+p+1}}{2j + 2p + 1}.$$

The above theorem will be derived as an immediate consequence of two lemmas that are given in the next section. In the third section the theorem will be applied to a few specific equations of form (1.1).

**2. Two lemmas.** First we shall derive a result for polynomials  $P(r, \theta, t)$  involving only even powers of  $t$ .

**LEMMA 1.** *Let*

$$P(r, \theta, t) = a_0(r, \theta) + a_1(r, \theta)t^2 + \dots + a_n(r, \theta)t^{2n},$$

where the  $a_j(r, \theta)$  are functions of class  $C'$  in  $r$  and  $\theta$ . Then the function

$$(2.1) \quad u_p(r, \theta) = \int_{-1}^1 e^{P(r, \theta, t)} (1 - t^2)^{p-1/2} (re^{i\theta}/2)^p dt$$

satisfies the recurrence formula

$$(2.2) \quad \partial u_p / \partial r = [p/r + P_r(r, \theta, (1 - T)^{1/2})] u_p,$$

where  $T$  is the operator such that,  $T$  acting  $k$  times upon  $u_p$ ,

$$(2.3) \quad T^k u_p = (2/re^{i\theta})^k u_{p+k}$$

for  $k=0, 1, \dots$ .

**PROOF.** From (2.1) we obtain by differentiating with respect to  $r$ :

$$(2.4) \quad \frac{\partial u_p}{\partial r} = \frac{p}{r} u_p + \int_{-1}^1 e^P \left[ \frac{\partial a_0}{\partial r} + \frac{\partial a_1}{\partial r} t^2 + \dots + \frac{\partial a_n}{\partial r} t^{2n} \right] (1 - t^2)^{p-1/2} \left( \frac{re^{i\theta}}{2} \right) dt.$$

To evaluate the latter integral, let us note that

$$\begin{aligned} \int_{-1}^1 t^{2m} e^{P(1-t^2)^{p-1/2}} (re^{i\theta}/2)^p dt &= \int_{-1}^1 [1 - (1-t^2)]^m e^{P(1-t^2)^{p-1/2}} (re^{i\theta}/2)^p dt \\ &= \sum_{k=0}^m (-1)^k C_{m,k} \int_{-1}^1 e^{P(1-t^2)^{k+p-1/2}} (re^{i\theta}/2)^p dt \\ &= \sum_{k=0}^m (-1)^k C_{m,k} (2/re^{i\theta})^p u_{p+k}. \end{aligned}$$

Hence,

$$(2.5) \quad \int_{-1}^1 t^{2m} e^{P(1-t^2)^{p-1/2}} (re^{i\theta}/2)^p dt = (1-T)^m u_p.$$

Substituting now from (2.5) into (2.4), we find

$$\begin{aligned} \frac{\partial u_p}{\partial r} &= \frac{p}{r} u_p + \frac{\partial a_0}{\partial r} u_p + \frac{\partial a_1}{\partial r} (1-T)u_p + \dots + \frac{\partial a_n}{\partial r} (1-T)^n u_p \\ &= [p/r + P_r(r, \theta, (1-T)^{1/2})] u_p, \end{aligned}$$

as was to be proved.

The corresponding result for a polynomial that involves only odd powers of  $t$  may be stated as follows.

LEMMA 2. Let  $Q(r, \theta, t) \equiv a_0(r, \theta)t + a_1(r, \theta)t^3 + \dots + a_n(r, \theta)t^{2n+1} \equiv tP(r, \theta, t)$ , where the  $a_j(r, \theta)$  are functions of class  $C'$  in  $r$  and  $\theta$ . Then the functions  $u_p(r, \theta)$  defined by (2.1) satisfy the recurrence formula:

$$(2.6) \quad \begin{aligned} \partial u_p / \partial r &= (p/r) u_p + \left\{ Q_t(r, \theta, (1-T)^{1/2}) \right. \\ &\quad \left. \cdot \int_0^{T^{1/2}} t^{2p} P_r(r, \theta, (1-t)^{1/2}) dt \right\} T^{-p+1/2} u_p, \end{aligned}$$

where  $T$  is the operator defined by equation (2.3).

PROOF. In place of (2.4), we now have

$$(2.7) \quad \begin{aligned} \frac{\partial u_p}{\partial r} &= \frac{p}{r} u_p + \int_{-1}^1 e^{Q} \left[ \frac{\partial a_0}{\partial r} t + \frac{\partial a_1}{\partial r} t^3 + \dots \right. \\ &\quad \left. + \frac{\partial a_n}{\partial r} t^{2n+1} \right] (1-t^2)^{p-1/2} \left( \frac{re^{i\theta}}{2} \right)^p dt. \end{aligned}$$

In order to evaluate the latter integral, let us first integrate by parts:

$$\begin{aligned}
 & \int_{-1}^1 t e^Q (1 - t^2)^{p-1/2} (r e^{i\theta} / 2)^p dt \\
 &= -\frac{1}{2} \int_{-1}^1 e^Q (1 - t^2)^{p-1/2} (r e^{i\theta} / 2)^p d(1 - t^2) \\
 &= (1/(2p + 1)) \int_{-1}^1 e^Q (\partial Q / \partial t) (1 - t^2)^{p+1/2} (r e^{i\theta} / 2)^p dt \\
 &= (1/(2p + 1)) \int_{-1}^1 e^Q [a_0 + 3a_1 t^2 + \dots \\
 &\qquad\qquad\qquad + (2n + 1)a_n t^{2n}] (1 - t^2)^{p+1/2} (r e^{i\theta} / 2)^p dt.
 \end{aligned}$$

Hence, by equation (2.5),

$$\begin{aligned}
 \int_{-1}^1 t e^Q (1 - t^2)^{p-1/2} (r e^{i\theta} / 2)^p dt &= (1/(2p + 1)) Q_i(r, \theta, (1 - T)^{1/2}) T u_p \\
 &= Q_i(r, \theta, (1 - T)^{1/2}) \left( \int_0^{T^{1/2}} t^{2p} dt \right) T^{-p+1/2} u_p.
 \end{aligned}$$

Let us then assume that the formula

$$\begin{aligned}
 (2.8) \quad & \int_{-1}^1 t^{2m+1} e^Q (1 - t^2)^{p-1/2} (r e^{i\theta} / 2)^p dt \\
 &= Q_i(r, \theta, (1 - T)^{1/2}) \left( \int_0^{T^{1/2}} t^{2p} (1 - t^2)^m dt \right) T^{-p+1/2} u_p
 \end{aligned}$$

has already been verified for  $m = 0, 1, 2, \dots, N$  and proceed to verify the formula for  $m = N + 1$ , as follows.

$$\begin{aligned}
 & \int_{-1}^1 t^{2N+3} e^Q (1 - t^2)^{p-1/2} (r e^{i\theta} / 2)^p dt \\
 &= \int_{-1}^1 t^{2N+1} e^Q (1 - t^2)^{p-1/2} (r e^{i\theta} / 2)^p dt \\
 &\quad - \int_{-1}^1 t^{2N+1} e^Q (1 - t^2)^{p+1/2} (r e^{i\theta} / 2)^p dt \\
 &= Q_i(r, \theta, (1 - T)^{1/2}) \left\{ \left[ \int_0^{T^{1/2}} t^{2p} (1 - t^2)^N dt \right] T^{-p+1/2} u_p \right. \\
 &\quad \left. - \left[ \int_0^{T^{1/2}} t^{2p+2} (1 - t^2)^N dt \right] (2/r e^{i\theta}) T^{-p-1/2} u_{p+1} \right\} \\
 &= Q_i(r, \theta, (1 - T)^{1/2}) \left( \int_0^{T^{1/2}} t^{2p} (1 - t^2)^{N+1} dt \right) T^{-p+1/2} u_p.
 \end{aligned}$$

Thus, formula (2.8) has been established by mathematical induction.

We now substitute from formula (2.8) into expression (2.7), thus obtaining

$$\begin{aligned} \frac{\partial u_p}{\partial r} &= \frac{\dot{p}}{r} u_p + \left( Q_t(r, \theta, (1-T)^{1/2}) \right. \\ &\quad \cdot \left. \int_0^{T^{1/2}} t^{2p} \left\{ \frac{\partial a_0}{\partial r} + \frac{\partial a_1}{\partial r} (1-t^2) + \dots + \frac{\partial a_n}{\partial r} (1-t^2)^n \right\} T^{-p+1/2} u_p \right) \\ &= \frac{\dot{p}}{r} u_p + Q_t(r, \theta, (1-T)^{1/2}) \left\{ \int_0^{T^{1/2}} t^{2p} P_r(r, \theta, (1-t)^{1/2}) \right\} T^{-p+1/2} u_p, \end{aligned}$$

as was to be proved.

PROOF OF THEOREM 1. Formula (2.2) may be reduced to formula (1.10) if, using (1.8) and (2.3), we set

$$\begin{aligned} P_r(r, \theta, (1-T)^{1/2}) u_p &= \sum_{j=0}^n \frac{\partial a_j}{\partial r} (1-T)^j u_p = \sum_{q=0}^n \alpha_q \left( \frac{re^{i\theta}}{2} \right)^q T^q u_p \\ &= \sum_{q=0}^n \alpha_q u_{p+q}. \end{aligned}$$

To reduce formula (2.6) to (1.11), let us set

$$Q_t(r, \theta, (1-T)^{1/2}) = \sum_{j=0}^n (2j+1) a_j (1-T)^j = \sum_{j=0}^n \beta_j \left( \frac{re^{i\theta}}{2} \right)^j T^j$$

with the  $\beta_j$  defined as in formula (1.9). Since

$$P_r(r, \theta, (1-t^2)^{1/2}) = \sum_{j=0}^n \alpha_j \left( \frac{re^{i\theta}}{2} \right)^j t^{2j},$$

we may write the second term of the left side of (2.6) as

$$\begin{aligned} \sum_{k=0}^n \beta_k \left( \frac{re^{i\theta}}{2} \right)^k T^k \left[ \int_0^{T^{1/2}} t^{2p} \sum_{j=0}^n \alpha_j \left( \frac{re^{i\theta}}{2} \right)^j t^{2j} dt \right] T^{-p+1/2} u_p \\ = \sum_{j,k=0}^n \frac{\alpha_j \beta_k}{2p+2j+1} \left( \frac{re^{i\theta}}{2} \right)^{j+k} T^{k+j+1} u_p = \frac{2}{re^{i\theta}} \sum_{j,k=0}^n \frac{\alpha_j \beta_k u_{p+j+k+1}}{2p+2j+1}. \end{aligned}$$

Thus the proof of our main theorem is completed.

**3. Examples.** Let us first verify that the recurrence relation (2.6) is a generalization of that for Bessel's functions as given in formula (1.7). Here  $Q(r, \theta, t) = rti$  and thus (2.6) becomes

$$\begin{aligned}
 (3.1) \quad \partial u_p / \partial r &= (p/r)u_p + ri \left[ \int_0^{T^{1/2}} t^{2p} i dt \right] T^{-p+1/2} u_p \\
 &= (p/r)u_p - (2e^{-i\theta} / (2p + 1)) u_{p+1}.
 \end{aligned}$$

If now we set

$$\begin{aligned}
 u_p &= (\pi^{1/2}/2)\Gamma(p + 1/2)J_p(r)e^{ip\theta}, \\
 u_{p+1} &= (\pi^{1/2}/2)(p + 1/2)\Gamma(p + 1/2)J_{p+1}(r)e^{i(p+1)\theta},
 \end{aligned}$$

formula (3.1) reduces at once to formula (1.7).

As our second example, let us consider the differential equation  $L(U)=0$  in which the expression  $F=c-ab-a_z \neq 0$  satisfies the two equations

$$(3.2) \quad F_z = 0, \quad 2F - a_z + b_z = 0.$$

As shown by Bergman,<sup>4</sup> one of the possible corresponding generating functions is  $E(z, \bar{z}, t) = \exp P(r, \theta, t) = \exp(a_0 + a_1 t^2)$ , where

$$(3.3) \quad a_0 = - \int_0^z a d\bar{z}, \quad a_1 = 2z \int_0^z F d\bar{z}.$$

According to our theorem, the recurrence relation satisfied by the basic solutions is in this case

$$(3.4) \quad (\partial u_p / \partial r) = (p/r)u_p + \alpha_0 u_p + \alpha_1 u_{p+1},$$

where

$$\alpha_0 = \partial a_0 / \partial r + \partial a_1 / \partial r = -ae^{-i\theta} + 2rF + 2e^{i\theta} \int_0^z F d\bar{z},$$

$$\alpha_1 = - (2/r e^{i\theta}) (\partial a_1 / \partial r) = -4e^{-i\theta} F - (4/r) \int_0^z F d\bar{z}.$$

A partial differential equation which satisfies conditions (3.2) is

$$(3.5) \quad U_{z\bar{z}} - 2(z + \bar{z})U_z + U = 0.$$

Here  $F(z) = 1$ ,  $a_0 = 0$ ,  $a_1 = 2r^2$ , and therefore in the recurrence relation (3.4)  $\alpha_0 = 4r$ , and  $\alpha_1 = -8e^{-i\theta}$ . Setting  $U = U^{(1)} + iU^{(2)}$ , we see that equation (3.5) is equivalent to the system of two partial differential equations

$$\begin{aligned}
 \Delta U^{(1)} - 8xU_x^{(1)} + 8xU_y^{(2)} + 4U^{(1)} &= 0, \\
 \Delta U^{(2)} - 8xU_x^{(2)} - 8xU_y^{(1)} + 4U^{(2)} &= 0,
 \end{aligned}$$

<sup>4</sup> See p. 1194, reference in footnote 1a.



and that recurrence relation (3.4) for  $u_p = u_p^{(1)} + iu_p^{(2)}$  is in this case equivalent to the system of recurrence relations

$$\begin{aligned} \partial u_p^{(1)} / \partial r &= (p/r)u_p^{(1)} + 4ru_p^{(1)} - 8u_{p+1}^{(1)} \cos \theta - 8u_{p+1}^{(2)} \sin \theta, \\ \partial u_p^{(2)} / \partial r &= (p/r)u_p^{(2)} + 4ru_p^{(2)} - 8u_{p+1}^{(2)} \cos \theta + 8u_{p+1}^{(1)} \sin \theta. \end{aligned}$$

Other examples of partial differential equations  $L(U) = 0$  for which log  $E$  is an even or odd polynomial in  $t$  may be found in the articles referred to in footnotes 1a and 3. For these differential equations also, a recurrence relation may be derived by use of Lemmas 1 and 2.

**4. Generalization.** By means of formulas (2.5) and (2.8), the theorem given in the introduction may be extended to partial differential equations of type (1.1) for which a generating function exists that has the form  $E = g \exp f$  with both  $f$  and  $g$  suitably chosen polynomials in  $t$ . The generalization may be stated as follows.

**THEOREM 2.** *Let  $L(U) = 0$  be a partial differential equation of type (1.1) for which a generating function  $E(z, \bar{z}, t)$  exists that has one of the forms*

- I.  $E(z, \bar{z}, t) = R(r, \theta, t) \exp P(r, \theta, t),$
- II.  $E(z, \bar{z}, t) = R(r, \theta, t) \exp tP(r, \theta, t),$
- III.  $E(z, \bar{z}, t) = tR(r, \theta, t) \exp tP(r, \theta, t),$

where

$$\begin{aligned} P(r, \theta, t) &= a_0(r, \theta) + a_1(r, \theta)t^2 + \dots + a_m(r, \theta)t^{2m}, \\ R(r, \theta, t) &= b_0(r, \theta) + b_1(r, \theta)t^2 + \dots + b_n(r, \theta)t^{2n}, \end{aligned}$$

and where the  $a_j(r, \theta)$  and  $b_j(r, \theta)$  are of class  $C^1$  in  $r$  and  $\theta$ . Let  $u_p(r, \theta)$  be the corresponding basic solutions and let  $R(\partial p / \partial r) = \sum_0^{m+n} c_j(r, \theta)t^{2j},$

$$\begin{aligned} \alpha_k &= (-1)^k \left( \frac{2}{re^{i\theta}} \right)^k \sum_{j=k}^m (2j+1) C_{j,k} a_j; \\ \beta_k &= (-1)^k \left( \frac{2}{re^{i\theta}} \right)^k \sum_{j=k}^n C_{j,k} \frac{\partial b_j}{\partial r}; \quad \gamma_k = (-1)^k \left( \frac{2}{re^{i\theta}} \right)^k \sum_{j=k}^{n+m} C_{j,k} c_j. \end{aligned}$$

Then the recurrence relation satisfied by these basic solutions is

$$\frac{\partial u_p}{\partial r} = \frac{p u_p}{r} + \sum_{k=0}^n \beta_k u_{p+k} + \sum_{k=0}^{n+m} \gamma_k u_{p+k}$$

if  $E$  has the form I;

$$\frac{\partial u_p}{\partial r} = \frac{p u_p}{r} + \sum_{k=0}^n \beta_k u_{p+k} + \left( \frac{2}{r e^{i\theta}} \right) \sum_{k=0}^m \sum_{\nu=0}^{m+n} \frac{\alpha_k \gamma_\nu u_{p+k+\nu+1}}{2p + 2\nu + 1}$$

if  $E$  has the form II; and

$$\begin{aligned} \frac{\partial u_p}{\partial r} = \frac{p u_p}{r} + \left( \frac{2}{r e^{i\theta}} \right) \sum_{k=0}^m \sum_{\nu=0}^n \left( \frac{\alpha_k \beta_\nu}{2p + 2\nu + 1} \right) u_{p+k+\nu+1} \\ + \gamma_0 u_p + \sum_{k=1}^{m+n} \left( \gamma_k - \frac{2}{r e^{i\theta}} \gamma_{k-1} \right) u_{p+k} \end{aligned}$$

if  $E$  has the form III.

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## ON GAUSS' AND TCHEBYCHEFF'S QUADRATURE FORMULAS

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The well known Gauss' Quadrature Formula

$$(1) \quad \int_{-\infty}^{\infty} G_k(x) d\psi(x) = \sum_{i=1}^n \rho_i^{(n)} G_k(\xi_i^{(n)})$$

is valid for every polynomial  $G_k(x)$ , of degree  $k \leq 2n-1$ , the  $\{\xi_i^{(n)}\}$  being the roots of the polynomial  $P_n(x)$ , orthogonal with respect to the distribution  $d\psi(x)$  ( $i=1, 2, \dots, n; n=1, 2, \dots$ ).<sup>1</sup> If the sequence  $\{P_n(x)\}$  is that of Tchebycheff (trigonometric) polynomials, then the Christoffel numbers  $\rho_i^{(n)}$ ,  $i=1, 2, \dots, n$ , are equal, and the two quadrature formulas of Gauss and Tchebycheff coincide:

$$(2) \quad \int_{-\infty}^{\infty} G_k(x) d\psi(x) = \rho_n \sum_{i=1}^n G_k(\xi_i^{(n)}), \quad k \leq 2n-1; n=1, 2, \dots$$

The converse—that this is the only case of coincidence of these formulas—was proved by R. P. Bailey [1a] and, under more restrictive conditions, by Krawtchouk [1b] (cf. also [2]).<sup>2</sup>

We shall give here four distinct proofs of this statement, without imposing any restrictions on  $\psi(x)$ .

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<sup>1</sup>  $\psi(x)$  is a bounded non-decreasing function, with infinitely many points of increase, for which all moments exist:  $c_n = \int_{-\infty}^{\infty} x^n d\psi(x); n=0, 1, 2, \dots$

<sup>2</sup> Numbers in brackets refer to the bibliography at the end of the paper.