ON GAUSS’ AND TCHEBYCHEFF’S QUADRATURE FORMULAS

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The well known Gauss’ Quadrature Formula

\[ \int_{-\infty}^{\infty} G_k(x) d\psi(x) = \sum_{i=1}^{n} G_k(\xi_i^{(n)}) \]

is valid for every polynomial \( G_k(x) \), of degree \( k \leq 2n - 1 \), the \( \{\xi_i^{(n)}\} \)
being the roots of the polynomial \( P_n(x) \), orthogonal with respect to the distribution \( d\psi(x) \) \( (i = 1, 2, \ldots, n; n = 1, 2, \ldots) \).\(^1\) If the sequence \( \{P_n(x)\} \) is that of Tchebycheff (trigonometric) polynomials, then the Christoffel numbers \( \rho_i^{(n)}, i = 1, 2, \ldots, n \), are equal, and the two quadrature formulas of Gauss and Tchebycheff coincide:

\[ \int_{-\infty}^{\infty} G_k(x) d\psi(x) = \rho_n \sum_{i=1}^{n} G_k(\xi_i^{(n)}), \quad k \leq 2n - 1; n = 1, 2, \ldots. \]

The converse—that this is the only case of coincidence of these formulas—was proved by R. P. Bailey [1a] and, under more restrictive conditions, by Krawtchouk [1b] (cf. also [2]).\(^2\)

We shall give here four distinct proofs of this statement, without imposing any restrictions on \( \psi(x) \).

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\(^1\) \( \psi(x) \) is a bounded non-decreasing function, with infinitely many points of increase, for which all moments exist: \( c_n = \int_{-\infty}^{\infty} x^n d\psi(x); n = 0, 1, 2, \ldots. \)

\(^2\) Numbers in brackets refer to the bibliography at the end of the paper.
Consider the sequence of complex numbers \( \{c_n\} \), \( n = 0, 1, 2, \ldots \), subject to the conditions
\[
(3) \quad \Delta_{n+1} \equiv |c_{i+k} - c_i|^{n+k} \neq 0; \quad n = 0, 1, 2, \ldots.
\]
Consider also the Stieltjes linear functional \( \sigma \), with
\[
(4) \quad \sigma(x^k) = c_k; \quad k = 0, 1, 2, \ldots,
\]
and the polynomials \( \{P_n(x)\} \), \( n = 0, 1, 2, \ldots \), orthogonal relative to the sequence \( \{c_n\} \), that is \([3, 4]\)
\[
(5) \quad \sigma \{P_n(x)P_m(x)\} = \begin{cases} 0, & m \neq n, \\ h_n \equiv \Delta_{n+1}/\Delta_n \neq 0, & m = n. \end{cases}
\]
Let \( \{\xi_i^{(n)}\} \), \( i = 1, 2, \ldots, n \), be the roots, distinct or not, of \( P_n(x) \), \( n = 1, 2, \ldots \).

I. The first method of proving our statement consists in proving the following theorem.

**Theorem.** From the validity of the formula
\[
(6) \quad \rho_n \sum_{i=1}^{n} (\xi_i^{(n)})^k = c_k, \quad n \geq \left[\frac{k+1}{2}\right],
\]
for \( k \leq 2 \), it follows that the \( \{P_n(x)\} \) are the Tchebycheff polynomials, so that (6) holds for all integral \( k \).

Introduce the mean of order \( \nu \) of the numbers \( \{\xi_i^{(n)}\} \), \( i = 1, 2, \ldots, n \):
\[
(7) \quad \mu_{\nu}^{(n)} = \left[\frac{1}{n} \sum_{i=1}^{n} (\xi_i^{(n)})^\nu\right]^{1/\nu}, \quad \nu = 1, 2, \ldots; n = 1, 2, \ldots.
\]
On equating the coefficients of \( x^{n-1} \) and \( x^{n-2} \) on both sides of the recurrence relation
\[
(8) \quad P_n(x) = (x - \alpha_n)P_{n-1}(x) - \lambda_n P_{n-2}(x),
\]
\( \lambda_n \neq 0, \quad P_{-1} \equiv 0, \quad n = 1, 2, \ldots, \)
we get
\[
(9) \quad \delta_1^{(n)} = - \sum_{k=1}^{n} \alpha_k, \quad \delta_2^{(n)} = - \sum_{k=2}^{n} \lambda_k + \sum_{r=2}^{n} \alpha_r \sum_{k=1}^{r-1} \alpha_k,
\]
where we let
\[
(9') \quad P_n(x) = x^n + \delta_1^{(n)} x^{n-1} + \delta_2^{(n)} x^{n-2} + \cdots + \delta_n^{(n)}.
\]
On putting \( k = 0, 1, 2 \) in (6) we obtain

\[
\rho_n = c_0/n; \quad \mu_1^{(n)} = c_1/c_0, \quad n \geq 1; \quad \mu_2^{(n)} = c_2/c_0, \quad n \geq 2.
\]

Since, by (9) and (9'),

\[
n\mu_1^{(n)} = \delta_1^{(n)} = \sum_{k=1}^{n} \alpha_k,
\]

\[
n(\mu_2^{(n)})^2 = (\delta_1^{(n)})^2 - 2\delta_2^{(n)} = \left( \sum_{k=1}^{n} \alpha_k \right)^2 + 2 \sum_{k=1}^{n} \lambda_k - 2 \sum_{r=2}^{n} \sum_{k=1}^{r-1} \alpha_k,
\]

we find:

\[
\alpha_1 = (\alpha_1 + \alpha_2)/2 = (\alpha_1 + \alpha_2 + \alpha_3)/3 = \cdots,
\]

whence we have the fundamental result:

\[
\alpha_1 = \alpha_2 = \cdots = \alpha,
\]

and similarly,

\[
\lambda_3 = \lambda_4 = \cdots = \lambda_2/2 = \lambda.
\]

Here \( \alpha \) and \( \lambda \) are independent of \( n \). The solution of (8) under these conditions is

\[
P_n(x) = \left[ (x - \alpha + ((x - \alpha)^2 - 4\lambda)^{1/2})^n
\right. \\
\left. + (x - \alpha - ((x - \alpha)^2 - 4\lambda)^{1/2})^n/2^n, \quad n = 1, 2, \cdots ,
\]

which shows that the \( \{ P_n(x) \} \) are the Tchebycheff polynomials, whence the validity of (6) for \( k = 3, 4, \cdots \) follows.

We have found incidentally the following property of the Tchebycheff polynomials.

**Corollary.** If the arithmetic mean and the root-mean-square of the roots \( \{ \xi_i^{(n)} \} \), \( i = 1, 2, \cdots , n \), of the orthogonal polynomials \( P_n(x) \), \( n = 1, 2, \cdots \), do not depend on \( n \), the means of all orders \( v \leq 2n - 1 \) possess the same property, and the \( \{ P_n(x) \} \) are the Tchebycheff polynomials (13).

II. Define

\[
R_{n-1}(y) = \sigma \left\{ (P_n(y) - P_n(x))/(y - x) \right\},
\]

a polynomial of degree \( n - 1; \ n = 1, 2, \cdots \). Our second proof consists in showing that (6) implies

\[
R_{n-1}(y) \equiv (c_0/n)P'_n(y), \quad n \geq 2.
\]
From (6), wherein $\rho_n$ necessarily equals $c_0/n$, we get, by virtue of (4),
\[
\sigma(x^k) = \frac{c_0}{n} \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} (\xi_i^{(n)})^k; \quad \sigma(G_h(x)) = \frac{c_0}{n} \sum_{i=1}^{n} G_h(\xi_i^{(n)}), \quad n \geq \frac{k+1}{2}.
\]
The desired relation
\[
R_{n-1}(y) = \frac{c_0}{n} \sum_{i=1}^{n} \frac{P_n(y) - P_n(\xi_i^{(n)})}{y - \xi_i^{(n)}} = \frac{c_0}{n} P_n(y) \sum_{i=1}^{n} \frac{1}{y - \xi_i^{(n)}} = \frac{c_0}{n} P_n'(y)
\]
follows. Now there are different methods of showing that (14) implies (13).

II. This statement is proved in a note [2] as a particular case of more general theorems.

II. Introduce the polynomials $\{P_n^{(i)}(x)\}, i = 1, 2, \cdots, n$—the denominators of the convergents of order $n$ of the continued fractions
\[
\lambda_i - \lambda_{i+1} - \cdots, \quad i = 1, 2, \cdots.
\]
We have $P_n^{(1)}(x) \equiv P_n(x); P_n^{(2)}(x) \equiv (1/c_0) R_n(x), n = 0, 1, 2, \cdots.$

It is easy to show that
\[
P_n(x) = (x - \alpha_1) P_{n-1}^{(2)}(x) - \lambda_2 P_{n-2}^{(3)}(x), \quad n = 2, 3, \cdots.
\]
On the other hand
\[
R_{n-1}(x) = (x - \alpha_n) R_{n-2}(x) - \lambda_n R_{n-3}(x), \quad n = 3, 4, \cdots.
\]
Using (8), (14) and (17), we find that
\[
P_{n-1}(x) = (x - \alpha_n) P_{n-2}^{(2)}(x) - 2\lambda_n P_{n-3}^{(3)}(x), \quad n = 3, 4, \cdots,
\]
which gives, in conjunction with (16),
\[
(\alpha_1 - \alpha_n) P_{n-2}^{(2)}(x) = 2\lambda_n P_{n-3}^{(3)}(x) - \lambda_2 P_{n-3}^{(3)}(x), \quad n = 3, 4, \cdots.
\]
Hence
\[
\alpha_3 = \alpha_4 = \cdots = \alpha_1 = \alpha; \quad \lambda_3 = \lambda_4 = \cdots = \lambda_2/2 = \lambda;
\]
\[
P_{n-3}^{(2)}(x) \equiv P_{n-3}^{(3)}(x).
\]
This identity, for $n = 4$, gives $\alpha_3 = \alpha$, and thus we have arrived again at (12, 12').

II. On putting

They were introduced by Stieltjes [5]; cf. also Perron [6].
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\[ R_n(y) = c_0(y^n + d_1^{(n)} y^{n-1} + \cdots + d_n^{(n)}), \]

we find from (17) the relations

\[ d_1^{(n-1)} = - \sum_{k=2}^{n} \alpha_k, \quad d_2^{(n-1)} = \sum_{k=3}^{n} \alpha_k \sum_{r=2}^{k-1} \alpha_r - \sum_{k=3}^{n} \lambda_k \]

analogous to (9). From (14) we find

\[ nd_1^{(n-1)} = (n - 1)\delta_1^{(n)}, \]

whence again \(\alpha_1 = \alpha_2 = \cdots = \alpha;\) and in the same way, the condition

\[ nd_2^{(n-1)} = (n - 2)\delta_2^{(n)} \]

again implies \(\lambda_3 = \lambda_4 = \cdots = \lambda_2/2 = \lambda.\)

We have found incidentally the following property of the Tchebycheff polynomials.

**Corollary.** If the three highest coefficients of the polynomials \(\{nR_{n-1}(x)\}\) and \(\{c_0P_n'(x)\}\), \(n=1, 2, \cdots,\) coincide, then these polynomials are identical, and the \(\{P_n(x)\}\) are the Tchebycheff polynomials (13).

**Bibliography**


4. ——— *On polynomials orthogonal with regard to a given sequence of numbers*, Transactions of the Kharkoff Mathematical Society vol. 17 (1940) pp. 3–18.


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