# POWERS OF HOMEOMORPHISMS WITH ALMOST PERIODIC PROPERTIES

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Let $X$ be a topological space (an "accessible space," a "1-space," or a "$T_1$-space" in the terminology of Fréchet, Kuratowski, or Alexandroff-Hopf, respectively) and let $f(X)=X$ be a homeomorphism. We use the following terminology, which was suggested by G. A. Hedlund and which is to be carefully distinguished from those terminologies used by Birkhoff, Ayres, Whyburn, and others. A point $x$ of $X$ is said to be *recurrent* under $f$ provided that to each neighborhood $U$ of $x$ there corresponds a positive integer $n$ such that $f^n(x)\in U$. The mapping $f$ is said to be *pointwise recurrent* provided that each point of $X$ is recurrent under $f$. A point $x$ of $X$ is said to be *almost periodic* under $f$ provided that to each neighborhood $U$ of $x$ there corresponds a monotone increasing sequence $n_1, n_2, \cdots$ of positive integers with the properties that the numbers $n_{i+1}-n_i$ $(i=1, 2, \cdots)$ are uniformly bounded and $f^{n_i}(x)\in U$ $(i=1, 2, \cdots)$. The mapping $f$ is said to be *pointwise almost periodic* provided each point of $X$ is almost periodic under $f$. Following Birkhoff [1, p. 198],[2] a subset $Y$ of $X$ is said to be *minimal* under $f$ provided that $Y$ is nonvacuous, closed and invariant under $f$, that is, $f(Y)=Y$, and furthermore $Y$ does not contain a proper subset with these properties. For $x\in X$, the set $\sum_{n=-\infty}^{+\infty} f^n(x)$ is called the *orbit* of $x$ under $f$ and the set $\sum_{n=0}^{+\infty} f^n(x)$ is called the *semi-orbit* of $x$ under $f$. A *decomposition* of $X$ is a collection of nonvacuous pairwise disjoint closed subsets of $X$ which fill up $X$.

THEOREM 1. *If $x\in X$ is recurrent under $f$, then $x$ is also recurrent under $f^n$ for every positive integer $n$.*

PROOF. We make use of an induction. The theorem is true for $n=1$. Let $m$ be any positive integer. Assume the theorem is true for $n\leqq m$. We now show the theorem is true for $n=m+1=k$.

We may suppose without loss of generality that $X$ is the closure of the semi-orbit of $x$ under $f$, for this set is invariant under $f$. Define $X_i$ $(i=0, 1, \cdots, k)$ to be the closure of the semi-orbit of $f^i(x)$ under $f^k$. It is readily verified that $f(X_i)=X_{i+1}$ $(i=0, 1, \cdots, k-1)$,

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

$f^k(X_i)\subset X_i$ ($i=0, 1, \cdots, k$), and $X=\sum_{i=0}^{k-1}X_i$. We may suppose that $x\in\sum_{i=1}^{k-1}X_i$, for otherwise the conclusion follows. Let $p$, $1\leqq p\leqq k-1$, be the smallest integer such that $x\in X_p$. Then, the semi-orbit of $x$ under $f^k$ is contained in $X_p$. Hence, $X_0\subset X_p=f^p(X_0)$ and $p$ is the smallest positive integer such that $X_0\subset f^p(X_0)$. Since $f^k(X_0)\subset X_0$, there exists a smallest positive integer $t$ such that $f^t(X_0)\subset X_0$. Now $f^t(X_0)\subset X_0\subset f^p(X_0)$. Since $f^{t-p}(X_0)\subset X_0$ and $X_0\subset f^{p-t}(X_0)$, $p$ is neither less than nor greater than $t$. Hence, $p=t$ and $f^p(X_0)=X_0$. Write $k=pq+r$, $0\leqq r<p$, where $q$ and $r$ are integers. Now

$$X_0\supset f^k(X_0) = f^r(f^{pq}(X_0)) = f^r(X_0).$$

Thus, $r=0$ and $k=pq$. If $p=1$, then $X_0=X_1=\cdots=X_k$ whence $x\in X_k$ and the conclusion follows. We may suppose, therefore, that $p>1$. Now $p\leqq m$ and $q\leqq m$. By the induction assumption, $x$ is recurrent under $f^p$ and, applying the induction assumption to $f^p$, $x$ is recurrent under $(f^p)^q=f^k$.

COROLLARY 1. *Every positive power of a pointwise recurrent homeomorphism on a topological space is itself pointwise recurrent.*

Theorem 1 can be used to provide a different proof of the following theorem, due to Birkhoff and Smith [2, p. 358, Theorem 3].

THEOREM. *If $X$ is a compact metric space and if $f(X)=X$ is a homeomorphism, then for every nonzero integer $n$ the central orbits under $f^n$ are identical with the central orbits under $f$.*

This follows from Theorem 1 and the result, due to Birkhoff and Smith [2, p. 353, Theorem 2], that the sum of the central orbits under a homeomorphism $h$ on a compact metric space is characterized as the closure of the set of points recurrent under both $h$ and $h^{-1}$. Although their results on central orbits [2, pp. 350–355, 356–360] are stated for closed surfaces, their proofs are actually valid for compact metric spaces.

THEOREM 2. *If $X$ is a compact connected metric space and if the recurrent points are dense in $X$, then every recurrent cut point $x$ of $X$ is periodic.*

PROOF. Express $X=A+B$, where $A$ and $B$ are nondegenerate continua such that $A\cdot B=x$. By a theorem due to Kelley [4, p. 194, Theorem 3.4] there exists an $F$-set (that is, either a simple link, or cut point, or end point) $F$ of $X$ such that $f(F)=F$. (For properties of simple links, see Whyburn [5, pp. 64–65].) Now $F$ is contained in either $A$ or $B$, say $A$, and $A\cdot f^n(A)\neq\Lambda$ for every integer $n$. Since some point of $B-x$ is recurrent under $f$, there exists a positive integer $n$ such that $B\cdot f^n(B)\neq\Lambda$. By Theorem 1, $x$ is recurrent under $f^n$. Applying a lemma due to Whyburn [5, p. 247, Lemma 4.21], it follows that $f^n(x)=x$. The proof is completed.

Theorem 2 and its proof are partial generalizations of Whyburn [5, p. 248, Theorem 4.6], but the original conclusion—that the mapping is elementwise periodic on all simple links—is no longer valid without semi-local connectedness, even though the mapping be regularly almost periodic in the sense of Whyburn [5, p. 250]. Theorem 1, however, may be used as an aid in the proof of the cited theorem.

*Remarks.* 1. If the subset $Y$ of $X$ is minimal under $f$, then $Y$ is minimal also under $f^{-1}$. 2. A nonvacuous subset $Y$ of $X$ is minimal if and only if the closure of the orbit of every point of $Y$ is $Y$. 3. The collection of sets minimal under $f$ is a decomposition of $X$ if and only if the closure of the orbits under $f$ is a decomposition of $X$; and, in either case, these two collections coincide. In other words, $f$ *gives a minimal-set decomposition* if and only if $f$ *gives an orbit-closure decomposition.*

THEOREM 3. *If $X$ is minimal under $f$ but not under $f^k$, where $k$ is a nonzero integer, then there exists an integer $n$, $n>1$, such that $n$ divides $|k|$ and $f^n$ gives a finite minimal-set decomposition which contains exactly $n$ elements.*

PROOF. By Remark 1, it is sufficient to prove the theorem when $k$ is positive. There exists a point $x$ of $X$ such that the orbit of $x$ under $f^k$ is not dense in $X$, by Remark 2. Define $X_i$ $(i=0, 1, \cdots, k-1)$ to be the closure of the orbit of $f^i(x)$ under $f^k$. Clearly, $f(X_i)=X_{i+1}$ $(i=0, 1, \cdots, k-2)$, $f(X_{k-1})=X_0$, and $f^k(X_i)=X_i$ $(i=0, 1, \cdots, k-1)$. Let $p$ be the maximum positive integer such that there exist integers $i_1, i_2, \cdots, i_p$ with the properties that $0\leqq i_1<i_2<\cdots<i_p\leqq k-1$ and $\prod_{j=1}^{p}X_{i_j}\neq\Lambda$. Choose integers $i_1, i_2, \cdots, i_p$ with these properties. Define $Y=\prod_{j=1}^{p}X_{i_j}$. Clearly, $f^k(Y)=Y$. Let $n$ be the smallest positive integer such that $f^n(Y)=Y$. Define $Y_j=f^j(Y)$ $(j=0, 1, \cdots, n-1)$. The sets $Y_j$ $(j=0, 1, \cdots, n-1)$ are closed and pairwise disjoint. Choose $y\in Y$. Then,

$$X=\overline{j\sum_{-\infty}^{+\infty}f^j(y)}=\overline{j\sum_{-\infty}^{+\infty}f^j(Y)}=\sum_{j=0}^{n-1}f^j(Y)=\sum_{j=0}^{n-1}Y_j.$$

Thus, $D\equiv[Y_j|j=0, 1, \cdots, n-1]$ is a decomposition of $X$.

We show $n>1$. Suppose $n=1$. Then, $X=Y_0=Y=X_{i_1}$, and thus

$X=f^{k-i_1}(X_{i_1})=f^k(X_0)=X_0$ whence the orbit of $x$ under $f^k$ is dense in $X$, contrary to the second statement of the proof.

We show that $n$ divides $k$. Write $k=qn+r$, $0\leqq r<n$, where $q$ and $r$ are integers. Then,

$$Y = f^k(Y) = f^r(f^{qn}(Y)) = f^r(Y).$$

Hence, $r=0$.

In order to show that each element $Y_j$ of $D$ is minimal under $f^n$, it is sufficient to observe that for $y\in Y_j$, the orbit of $y$ under $f$ is dense in $X$, $D$ is a decomposition of $X$ whose elements are invariant under $f^n$, and the subset of the orbit of $y$ under $f$ which is contained in $Y_j$ is actually the orbit of $y$ under $f^n$.

COROLLARY 2. *If $X$ is connected and minimal under $f$, then $X$ is also minimal under $f^n$ for every nonzero integer $n$.*

COROLLARY 3. *If $X$ has only finitely many, say $k$, components and if $X$ is minimal under $f$, then for every nonzero integer $n$ the mapping $f^n$ gives a finite minimal-set decomposition, the number of whose elements is the greatest common divisor of $k$ and $|n|$.*

If $k=1$, Corollary 3 reduces to Corollary 2. If $k>1$, Corollary 3 may be proved by first of all considering the case when $k=k'$ and $n=n'>0$ are relatively prime and then extending the result to $k=ak'$ and $n=an'$, where $a$ is any positive integer. Corollary 3 essentially combines Corollary 2 with a property of cyclic counting or, what is the same, a property of periodic orbits.

THEOREM 4. *If $X$ is minimal under $f$, then for every nonzero integer $n$ the mapping $f^n$ gives a finite minimal-set decomposition of $X$ into at most $|n|$ elements.*

PROOF. By Remark 1, it is sufficient to prove the theorem when $n$ is positive. We make use of an induction. The theorem is true for $n=1$. Let $m$ be any positive integer. Assume the theorem is true for $n\leqq m$. We now show the theorem is true for $n=m+1=k$.

If $X$ is minimal under $f^k$, the conclusion follows. Suppose now that $X$ is not minimal under $f^k$. By Theorem 3, there exist integers $p$ and $q$ such that $p>1$, $k=pq$, and $f^p$ gives a finite minimal-set decomposition $D$ of $X$ into exactly $p$ elements. Let $Y$ be any element of $D$. Now apply the induction assumption to $f^p(Y)=Y$ and $n=q\leqq m$. Thus $(f^p)^q=f^k$ gives a finite minimal-set decomposition of $Y$ into at most $q$ elements. Hence, $(f^p)^q=f^k$ gives a finite minimal-set decomposition of $X$ into at most $pq=k$ elements.

COROLLARY 4. *If the mapping $f$ gives an orbit-closure decomposition of $X$, then for every integer $n$ the mapping $f^n$ also gives an orbit-closure decomposition of $X$.*

PROOF. By virtue of Remark 3, it is sufficient to apply Theorem 4 to the elements of the orbit-closure decomposition given by $f$.

LEMMA 1. *If $X$ is a metric space and if $x \in X$ is almost periodic under $f$, then the closure $Y$ of the orbit of $x$ under $f$ is minimal under $f$.*

PROOF. Suppose $Y$ is not minimal. Then, there exists a nonvacuous closed invariant subset $Z$ of $Y$ such that $x \notin Z$. Choose $z \in Z$. Let $2\epsilon$ be the distance from $x$ to $Z$. There exists a positive integer $N$ such that in every set of $N$ consecutive positive integers appears an integer $n$ so that $\rho(x, f^n(x)) < \epsilon$, where $\rho$ is the metric in $X$. Choose $\delta > 0$ so small that $x' \in X$ with $\rho(z, x') < \delta$ implies $\rho(f^i(z), f^i(x')) < \epsilon$ $(i = 1, 2, \cdots, N)$. There exists an integer $p \geqq 0$ such that $\rho(z, f^p(x)) < \delta$. Also it is possible to find an integer $q$, $1 \leqq q \leqq N$, so that $\rho(x, f^{p+q}(x)) < \epsilon$. Furthermore, $\rho(f^q(z), f^{p+q}(x)) < \epsilon$. Hence, $\rho(x, f^q(z)) < 2\epsilon$ which is impossible because $f^q(z) \in Z$.

LEMMA 2. *If $X$ is a compact metric space and if $f$ gives an orbit-closure decomposition, then $f$ is pointwise almost periodic.*

PROOF. Suppose that some point $x$ of $X$ is not almost periodic. Then there exist a neighborhood $U$ of $x$ and a sequence $m_1, m_2, \cdots$ of positive integers such that $U \cdot \sum_{j=-i}^{+i} f^{m_i+j}(x) = \Lambda$ $(i = 1, 2, \cdots)$. We may suppose that the sequence $\{f^{m_i}(x)\}$ converges to some point, say $y$, of $X$. It is easy to show that the orbit of $y$ is contained in $X - U$. Hence, the closure of the orbit of $y$ is a proper subset of the closure of the orbit of $x$. This is impossible.

THEOREM 5. *If $X$ is a metric space, then in order that $f$ give an orbit-closure decomposition it is sufficient that $f$ be pointwise almost periodic; and in case $X$ is compact, this condition is also necessary.*

The proof follows easily from Lemmas 1 and 2 and Remark 3. Theorem 5 and Lemmas 1 and 2 are closely related to Hall and Kelley [**3**, p. 628, Theorem 4] and to Birkhoff [**1**, p. 199].

THEOREM 6. *Every power (including negative powers) of a pointwise almost periodic homeomorphism on a compact metric space is itself pointwise almost periodic.*

The proof follows readily from Theorem 5 and Corollary 4.

THEOREM 7. *If $X$ is a compact metric space and if $x \in X$ is almost* *periodic under f, then x is also almost periodic under $f^n$ for every integer n.*

The proof proceeds easily from Lemma 1, Remark 2, and Theorems 5 and 6.

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# SOME PROPERTIES OF SUMMABILITY. II[1]

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**1. Summability of bounded sequences.** It follows from a well known result of H. Steinhaus[2] that no regular matrix method of summability can be effective for (that is, assign a finite limit to) every element in the space $(m)$ of bounded sequences. The object of this note is to consider some questions suggested by this fact. The first of these may be formulated as follows. If $A$ is a given regular matrix method let $J_A$ denote the set of all $A$-summable bounded sequences. We then ask what are necessary and sufficient conditions on a subset $E$ of $(m)$ in order that there exist a regular $A$ such that $E \subset J_A$? In Theorem 1 below it is shown that the separability of $E$ is a sufficient condition. It seems unlikely that this condition is necessary although we have been unable to decide the question. It is clearly equivalent to the question of whether every $J_A$ is separable.

THEOREM 1. *Let $E$ be an arbitrary separable subset of $(m)$. Then every regular matrix $A = (a_{mk})$ contains a (necessarily regular) row-submatrix $B = (a_{m_i k})$ such that $E \subset J_B$.*

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[1] This note is in the nature of an appendix to the paper cited in footnote 4.

[2] H. Steinhaus, *Some remarks on the generalizations of the notion of limit* (in Polish), Prace Matematyczno-Fizyczne vol. 22 (1921) pp. 121–134. See also I. Schur, *Über lineare Transformationen in der Theorie der unendlichen Reihen*, J. Reine Angew. Math. vol. 151 (1921) pp. 79–111.