THE TRANSFORMATION OF ČECH

V. G. GROVE

1. Introduction. The purpose of this paper is to give a simple construction of the general transformation of Čech [1, p. 192].[1]

Let the differential equations of a surface S be written in the Fubini canonical form [2, p. 123]

$$x_{uu} = \theta_u x_u + \beta x_v + p x, \qquad x_{vv} = \gamma x_u + \theta_v x_v + q x, \qquad \theta = \log (\beta\gamma). \tag{1}$$

Let the differential equation defining a conjugate net N on S be written in the form

$$dv^2 - \lambda^2 du^2 = 0. \tag{2}$$

The ray and the associate ray intersect in *the canonical point* [3, p. 7] of N. The line joining the point x to the canonical point intersects the reciprocal of the Green-Fubini projective normal in a point whose coordinates are

$$(\beta/\lambda^2)x_u - \gamma\lambda^2 x_v. \tag{3}$$

We shall call this point *the conjugal point* of N at x.

2. Conjugal quadrics. Let the coordinates X of a point X be written in the form

$$X = x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv}.$$

Then with properly selected unit point, (x_1, x_2, x_3, x_4) are the coordinates of X referred to the tetrahedron (x, x_u, x_v, x_{uv}). The equation of the three-parameter family of quadrics each of which has second order contact [2, p. 142] with S at x is

$$x_2 x_3 + x_4(- x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4) = 0. \tag{4}$$

The equation of any plane through the conjugal point (3) is

$$x_1 - k(\gamma\lambda^2 x_2 + (\beta/\lambda^2)x_3) - 2l x_4 = 0. \tag{5}$$

We shall speak of this plane as *the conjugal plane* of N at x.

If we impose the condition that the polar plane of the covariant point (0, 0, 0, 1) with respect to the quadric (4) be the conjugal plane

Presented to the Society, November 27, 1943; received by the editors December 10, 1943.

[1] Numbers in brackets refer to the references cited at the end of the paper.

(5), we find that the equation of the quadric (4) assumes the form

$$(6) \qquad x_2x_3 + x_4[-x_1 + k(\gamma\lambda^2 x_2 + (\beta/\lambda^2)x_3) + lx_4] = 0.$$

We shall call the quadric so determined *a conjugal quadric* of S at x with respect to the conjugate net N. There is a two-parameter family of such quadrics associated with a given conjugate net. It follows from (6) that *a given conjugal quadric may also serve as the conjugal quadric of the associate conjugate net of N only if it is a quadric of Darboux*. Such a quadric serves as the conjugal quadric of any conjugate net.

Davis has defined [**3**, p. 12] a pencil of quadrics by demanding that the tangents to the curves of intersection of a quadric of the family (4) with S be apolar to the tangents to the curves of a conjugate net. *The quadric of Davis for the conjugate net defined by* (2) *is the conjugal quadric* (6) *with* $k=1$. The quadric of Davis for the associate conjugate net of N is the conjugal quadric (6) with $k=-1$. We may call this quadric the *conjugate quadric* of Davis.

If one demands that a quadric of the family (4) have third order contact with each of the curves of N, the quadric must be a conjugal quadric with $k=-1/3$. The quadric with $k=1/3$ plays the same role with the associate conjugate net.

It will be recalled [**2**, pp. 185, 187] that the lines joining the points whose general coordinates are

$$x_u - bx, \qquad x_v - ax,$$

wherein

$$(7) \quad a = (\theta_v + \lambda_v/\lambda + m\beta/\lambda^2)/2, \qquad b = (\theta_u - \lambda_u/\lambda + m\gamma\lambda^2)/2,$$

are respectively the flex-ray, ray, associate ray, principal join and associate principal join according as

$$m = 0, \, -1, \, 1, \, 5/3, \, -5/3.$$

The lines joining x to the point whose coordinates are $x_{uv}-a'x_u-b'x_v$ with a', b' defined by

$$(8) \quad a' = (\theta_v + \lambda_v/\lambda + n\beta/\lambda^2)/2, \quad b' = (\theta_v - \lambda_u/\lambda + n\gamma\lambda^2)/2$$

are respectively the axis, associate axis, cusp-axis, polar reciprocals of the principal join and associate principal join with respect to any quadric of Darboux according as $n=1, \, -1, \, 0, \, 5/3, \, -5/3$.

Imposing the condition that the lines determined by (7) be the polar lines of those defined by (8) with respect to the quadric (4) we find that quadric must be a conjugal quadric with

$$k = (n - m)/2. \tag{9}$$

It follows that the conjugal quadrics may be defined as quadrics having second order contact with S at x with respect to which canonical [**3**, p. 7] *lines of the first kind are reciprocal polars of canonical lines of the second kind.*

Using the various values of m and n for the well known canonical lines above we find interpretations for the conjugal quadrics for the particular values $k=0$, $\pm 1/3$, $\pm 1/2$, $\pm 5/6$, ± 1, $\pm 4/3$, $\pm 5/3$.

In particular *the quadric of Davis is a quadric having second order contact with S at x and for which the axis of N is the polar of the ray of N*. Thus a new characterization of this quadric is obtained.

By means of the first characterization of the conjugal quadric with $k=-1/3$, we may give characterizations of some of the canonical lines associated with the net N. In particular *the principal join of the curves of N may be characterized as the polar line of the axis of the net with respect to the conjugal quadric $k=-1/3$; or it is the reciprocal of the associate ray of N with respect to the quadric of Davis.*

3. **The transformation of Čech.** In this section we shall show how the conjugal quadrics (6) may be considered as inducing the general transformation of Čech.

The coordinates of any point ρ on the tangent to the curve defined by $dv-\lambda du=0$ of the net N may be written in the form

$$\rho = x_u + \lambda x_v + \mu x.$$

The polar plane of ρ with respect to a conjugal quadric (6) has local coordinates given by the formulas

$$u_1 = 0, \qquad u_2 = \lambda^2, \qquad u_3 = \lambda, \qquad u_4 = -\lambda\mu + k(\beta + \gamma\lambda^2). \tag{10}$$

If the local coordinates of ρ be written in the form $(x_1, x_2, x_3, 0)$, equations (10) assume the form

$$\begin{aligned} u_1 &= 0, \qquad u_2 = x_2 x_3^2, \qquad u_3 = x_2^2 x_3, \\ u_4 &= -x_1x_2x_3 + k(\beta x_2^3 + \gamma x_3^3) \end{aligned} \tag{11}$$

of general transformation of Čech. We shall say that *the conjugal quadrics* (6) *induce the general transformation of Čech.* In particular *the quadrics of Darboux induce the polarity of Lie*, $k=0$, and *the quadrics of Davis induce the correspondence of Segre*, $k=1$.

The associate quadric of Davis may be said to induce *the associate correspondence of Segre*, $k=-1$. *The conjugal quadric $k=-1/3$ induces the correspondence of Moutard.* We may call the transformation

(11) with $k=1/3$ *the associate correspondence of Moutard*. The quadric (6) with $k=1/3$ induces this latter correspondence. The other particular conjugal quadrics characterized in the previous section of course induce the corresponding transformations of Čech.

Lane has given [**2**, p. 203] certain properties of the general transformation of Čech. We call attention to one additional property. We may easily verify that the most general linear transformation which leaves the form of the equations of the family of conjugal quadrics (6) invariant is the transformation

$$\begin{gathered}\sigma x_1 = \bar{x}_1 - b\bar{x}_2 - a\bar{x}_3 - c\bar{x}_4, \\ \sigma x_2 = \bar{x}_2 - a\bar{x}_4, \qquad \sigma x_3 = \bar{x}_3 - b\bar{x}_4, \qquad \sigma x_4 = \bar{x}_4.\end{gathered} \tag{12}$$

The transformation (12) and the corresponding transformation in plane coordinates leave the form of each of the transformations of Čech (11) absolutely invariant. *It follows therefore from the form of* (12) *that the transformation of the form* (11) *obtained by using any R-harmonic line* [**4**, p. 584] *and its reciprocal in the definition of the conjugal quadrics is the transformation of Čech for the same value of* k.

References

1. E. Čech, *L'intorno di un punto d'una superficie dal punto di vista proiettivo*, Annali di Matematica (3) vol. 31 (1922) pp. 191–206.

2. E. P. Lane, *A treatise on projective differential geometry*, The University of Chicago Press, 1942.

3. W. M. Davis, *Contributions to the general theory of conjugate nets*, dissertation, Chicago, 1932.

4. V. G. Grove, *On canonical forms of differential equations*, Bull. Amer. Math. Soc. vol. 36 (1930) pp. 582–586.

Michigan State College