A NOTE ON RIESZ SUMMABILITY OF THE TYPE $e^{na}$

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Recently I proved the following result in the case $r = 2$ (Wang [4]).

Let $a^{(r)}_n$ be the $r$th Cesàro mean of the series $\sum_{n=0}^{\infty}a_n$. If $a^{(r)}_n - s = o(n^{-r\alpha})$ $0 < \alpha < 1$, as $n \to \infty$, where $r$ is a positive integer, and $a_n > -Kn^{\alpha-1}$, the series converges to sum $s$.

For the case $r = 1$ this result has been established by Boas [1]. His argument, however, does not seem to be applicable in any simple way to the general case.

The object of this note is to prove a theorem on Riesz summability of type $e^{na}$, and then to deduce the result above from a Tauberian theorem of Hardy [2].

Let us put $C_r(\omega) = a_0 e^{\omega\alpha} + \sum_{n<\omega} (e^{na} - e^{na})^r a_n$. A series $\sum_{n=0}^{\infty}a_n$ is said to be summable $(e^{na}, r)$ to the sum $s$ if

\[(1) \quad C_r(\omega) = se^{\omega\alpha} + o(e^{\omega\alpha}).\]

The result by Hardy which is to be called upon is the following: If the series $\sum_{n=0}^{\infty}a_n$, with terms $a_n \geq -Kn^{\alpha-1}$, $0 < \alpha < 1$, is summable $(e^{na}, r)$, it is convergent. We shall now prove the following theorem.

**Theorem.** If $a^{(r)}_n - s = o(n^{-r\alpha})$, $0 < \alpha < 1$, as $n \to \infty$, the series $\sum_{n=0}^{\infty}a_n$ is summable $(e^{na}, r)$ to the sum $s$, where $r > r/(1-\alpha)$.

To prove this let $\beta_n = (e^{na} - e^{na})^r$, $\Delta \beta_n = \beta_n - \beta_{n+1}$, $\Delta^{r+1} \beta_n = \Delta \beta_n - \Delta \beta_{n+1}$ and

\[s^{(r)}_n = \sum_{\nu=0}^{n} \binom{n-\nu+r}{n-\nu} a_{\nu}, \quad m = [\omega].\]

Then, by successive Abel’s transformations we have

\[C_r(\omega) = a_0 e^{\omega\alpha} + \sum_{n=1}^{m} \beta_n a_n = a_0 e^{\omega\alpha} + \sum_{n=1}^{m-1} s^{(r)}_n \Delta^{r+1} \beta_n + \sum_{i=0}^{r-1} s^{(i)}_{m-i} \Delta^{i} \beta_{m-i} - \sum_{i=0}^{r} s^{(i)}_0 \Delta^{i} \beta_1 = a_0 e^{\omega\alpha} + J_1 + J_2 - J_3.\]

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1 Numbers in brackets refer to the references listed at the end of the paper.
Since $\beta_{m-i} = (e^{\omega x} - e^{(m-i)x})r = O(e^{r\omega^2 r(a-1)})$, it follows that

$$\Delta^{(i)}\beta_{m-i} = O(e^{r\omega^2 r(a-1)})$$

for $0 \leq i \leq r$.

By a familiar theorem on Cesàro sums we get

$$s_{m-i}^{(i)} = O(\omega^r),$$

for $0 \leq i \leq r$,

and from this

$$J_2 = O(e^{r\omega^2 r(a-1)+r}) = O(e^{r\omega^2}).$$

Since $\Delta^i\beta_1 = O(e^{r\omega^2 r(a-1)})$, for $1 \leq i \leq r$, and $\beta_1 = (e^{\omega x} - e)^r$, $s_0^{(i)} = s_0 = a_0$ we get

$$J_3 = e^{r\omega^2}a_0 + o(e^{r\omega^2}).$$

By the hypothesis of the theorem we have

$$J_1 = \sum_{n=1}^{m-r-1} s_n^{(r)} \Delta^{r+1} \beta_n = s \sum_{n=1}^{m-r-1} \left( \frac{n+1}{r+1} \right) \Delta^{r+1} \beta_n$$

$$+ o \left( \sum_{n=1}^{m-r-1} n^{(1-a)} | \Delta^{r+1} \beta_n | \right).$$

It follows by mathematical induction that

$$\Delta^{r+1} \beta_n = (-1)^{r+1} \int_0^{n+1} dx_1 \int_{x_1}^{x_1+1} dx_2 \cdots \int_{x_r}^{x_r+1} B^{(r+1)}(x_{r+1}) dx_{r+1},$$

where

$$\beta^{(r+1)}(x) = \frac{d^{r+1}}{dx^{r+1}} \left\{ (e^{\omega x} - e)^r \right\}.$$
Hence by (5), (6), and (7)

\[ J_1 = se^{r\sigma} + o(e^{r\sigma}) + o \left( \sum_{i=1}^{r} \sum_{n=1}^{m-1} e^{(r-i)\sigma} e^{i\pi n (r-1)} \right) \]

\[ = se^{r\sigma} + o(e^{r\sigma}). \]

The proof of the theorem follows from (4), (2), (3), and (8).

I conclude by observing that the theorem is the best possible of its kind (Wang [4]).

REFERENCES


