SOME REMARKS ON CONNECTED SETS

P. ERDÖS

This note will consist of a few disconnected remarks on connected sets.

i. Swingle[1] raised the question whether the plane is the sum of c disjoint biconnected sets. The answer as we shall show is affirmative.

First we construct a biconnected set A with a dispersion point x such that any two points of $A-x$ can be separated. (The first such set was constructed by Wilder.[2] Our construction will be very similar to that of Burton Jones.[3])

Our biconnected set will contain the origin $\mathfrak{O}$, at most one other point on lines through the origin with irrational slopes, and no point other than the origin on lines with rational slopes. Further it will contain at least one point on every cut of the plane. It is easy to see that such a set exists. Every cut of the plane contains a closed subset which also cuts the plane, and the power of closed sets is known to be c. Let us well order the closed cuts $C_1, C_2, \cdots, C_\gamma, \cdots, \gamma<\Omega_\alpha$, where Ω_α is the least ordinal number of power c. We construct A as follows: $\mathfrak{O}$ belongs to A. We shall choose a point x_γ on C_γ and we shall have $A=\bigcup x_\gamma$. We shall determine x_γ by transfinite induction. Suppose we have already determined x_δ, $\delta<\gamma$, we determine x_γ as follows: if C_γ contains $\mathfrak{O}$ then $x_\gamma=\mathfrak{O}$. If C_γ does not contain $\mathfrak{O}$ then clearly C_γ has to intersect c lines through $\mathfrak{O}$. Therefore we can find a point $x_\gamma \in C_\gamma$ such that $(\mathfrak{O}, x_\gamma)$ has irrational slope and does not go through any other point x_δ, $\delta<\gamma$. (We denote by (a, b) the line through the points a and b.) This way we construct A. Clearly A is biconnected. First of all A is connected since it intersects every cut of the plane. Also any two points of $A-\mathfrak{O}$ can be separated since the two points x_1 and x_2, say, are on different irrational lines through the origin, and a rational line, which of course does not intersect $A-\mathfrak{O}$, will separate them. This completes the proof. The origin we call the center of A.

Now we shall split the plane into the sum of c such disjoint sets A_γ, $\gamma<\Omega_\alpha$, Ω_α the smallest ordinal of power c. Let us well order the points x_γ of the plane and the closed cuts C_γ of the plane. Our first step is to select x_1 as the center of A_1 and a suitable point of A_1 on C_1.

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[1] P. M. Swingle, Amer. J. Math. vol. 54 (1932) p. 532.

[2] R. L. Wilder, Bull. Amer. Math. Soc. vol. 33 (1927) p. 423.

[3] Burton Jones, Bull. Amer. Math. Soc. vol. 48 (1942) p. 115.

(By suitable we mean that it does not lie on any line with rational slope through x_1.) Our second step is to select the first point which has not yet been selected as center of A_2, then select suitable points on C_1 and C_2 for A_2 and on C_2 for A_1. Suppose that this construction has been already carried out for all $\delta<\gamma$. Then the γth step consists of the following construction: First we take the first x_κ which has not yet been chosen as center of A_γ. Then we choose suitable points on all the C_δ, $\delta\leqq\gamma$, for A_γ and suitable points on C_γ for all the A_δ, $\delta<\gamma$. it is easy to see that this construction can be carried out since before the γth step we have chosen less than c points x_j, and a cut C_γ if it does not go through the origin intersects c lines through the origin. (if the cut C goes through the origin of one or more of the A_γ's then of course we can choose the origin.) Thus the construction can be carried out for all ordinals $\gamma<\Omega_\alpha$. Hence we get c sets A_γ where $\bigcup A_\gamma$ is the whole plane, $A_{\gamma_1}\cap A_{\gamma_2}=0$ and the A_γ are all biconnected, which completes the proof of the theorem.

A simple modification of this proof would give the following result: Let $m\leqq c$ be any cardinal number greater than 2, then the plane can be expressed as the sum of m disjoint biconnected sets.[4]

ii. Let C be any connected set. Knaster and Kuratowski[5] proved that there exists a proper connected subset of C. (Single points, of course, do not count as connected sets.) Their proof is very simple. Let $p\in C$ be arbitrary. If $C-p$ is connected our result is proved. If $C-p$ is not connected it can be written in the form $U+V$ where U and V are separated. But then it is easy to see that both $U+p$ and $V+p$ are connected.

It seems likely that the following result is true: every connected set C contains a connected subset C' such that $C-C'$ has power c. I am unable to prove this. It is easy to see that if a set C does not satisfy this conjecture it has to be both biconnected and widely connected, thus must be rather pathological.

A. Stone[6] proved that every connected set C has a connected subset C' such that $C-C'$ is infinite. Proof: First we can assume that $C-p$ is connected for every $p\in C$, for if not then at least one of the sets $U+p$ or $V+p$ has a complement of power c. (U and V were defined in the previous paragraph.) Similarly for every sequence $p_1, p_2, \cdots, p_n$, $C-p_1-p_2-\cdots-p_n$ is connected. Choose $p_1, p_2, \cdots$

[4] Swingle proved (ibid.) that the Euclidean n space is the sum of r biconnected sets if r is any finite number not less than $n+1$, also that it is the sum of $\aleph_0$ biconnected sets. He also showed that n-space is not the sum of n or less biconnected sets.

[5] Knaster-Kuratovski, Fund. Math. vol. 2 (1921) p. 206.

[6] Arthur Stone, oral communication.

such that it will converge to a point $p \in C$. Then $C - \sum_{i=1}^{\infty} p_i$ is connected. For if $\sum_{i=1}^{\infty} p_i$ would separate C, a closed subset of it would also separate C. But a closed subset of $\sum_{i=1}^{\infty} p_i$ is finite, and so by assumption can not separate C, and this completes the proof.

In this connection it might be of some interest to construct a connected set A with the following properties: (1) no finite subset disconnects A, (2) every countable dense subset totally disconnects A. For our construction we shall have to use the continuum hypothesis.

First we need the following lemma: Let S_1 and S_2 be two countable disjoint sets. S_1 is everywhere dense in the plane. Then there exist countably many closed Jordan curves J_r such that $J_r \cap S_1$ is dense on J_r, $J_r \cap S_2 = 0$, and to any two points p and q there exists an r such that J_r separates p and q. The proof does not present any difficulties and can be omitted.

And now to our construction.[7] First we well order the countable dense subsets of the plane, $D_1, D_2, \cdots, D_\alpha, \cdots$ (their number is clearly $c = \aleph_1$; the continuum hypothesis has been assumed). Also we well order all the closed connected cuts of the plane C_δ, again clearly $\delta < \Omega_1$ (it is well known that every cut of the plane contains a closed connected cut). We shall now construct our set A by transfinite induction. Let Δ be any given countable dense set, put $\Delta \subset A$. Our first step is to define a countable subset P_1 of C_1. If $\Delta \cap C_1$ is infinite we put $P_1 = 0$, if $C_1 \cap \Delta$ is finite, we choose an arbitrary countably infinite $P_1 \subset C_1$ and put $P_1 \subset A$. Consider next the smallest α such that $D_\alpha \subset \Delta + P_1$. Put $S_1 = D_\alpha$ and $S_2 = \Delta + P_1 - D_\alpha$. Then by our lemma there exists a countable collection of closed Jordan curves $J_r^{(1)}$ such that $J_r^{(1)} \cap D_\alpha$ is dense on $J_r^{(1)}$ and $J_r^{(1)} \cap \Delta + P_1 - D_\alpha = 0$ and any two points p and q are separated by some $J_r^{(1)}$. Now we make the condition that no point of any P_β $(\beta > 1)$ should lie on any J_r^β. Suppose we carried out our construction for any $\gamma < \delta$, we shall show that we can carry it out for $\gamma = \delta$. Consider C_δ; if $C_\delta \cap \Delta + \sum_{\beta<\delta} P_\beta$ is infinite, then $P_\delta = 0$. If $C_\delta \cap \Delta + \sum_{\beta<\delta} P_\beta$ is finite, then we choose a countably infinite subset P_δ of C_δ, such that P_δ does not meet $\sum_{\lambda<\delta} \sum_{r=1}^{\infty} J_r^\lambda$. In other words P_δ does not meet any of countably many Jordan curves (we have assumed the continuum hypothesis). We have to prove that such a choice of P_δ is possible. By assumption $(\Delta + \sum_{\beta<\delta} P_\beta) \cap C_\delta$ is finite. For each of the Jordan curves J_r^λ (with $\lambda < \delta$) $J_r^\lambda \cap \Delta + \sum_{\beta<\delta} P_\beta$ is dense on J_r^λ, therefore $C_\delta \cap J_r^\lambda$ is nowhere dense on C_δ (for if not, since C_δ is closed, it would contain a portion of J_r^λ and therefore $C_\delta \cap (\Delta + \sum_{\beta<\delta} P_\beta)$ would be infinite, as $\Delta + \sum_{\beta<\delta} P_\beta$ is dense on J_r^λ).

[7] Our construction will be similar to Miller's construction of a biconnected set without a dispersion point. (Fund. Math. vol. 29 (1937) p. 123.)

Therefore $C_\delta \cap \sum_{\lambda<\delta} \sum_{r=1}^{\infty} J_r^\lambda$ is of first category on C_δ, and thus there are c points of C not belonging to $\sum_{\lambda<\delta} \sum_{r=1}^{\infty} J_r^\lambda$; therefore we can choose an infinite $P_\delta \subset C_\delta$, $P_\delta \cap \sum_{\lambda<\delta} \sum_{r=1}^{\infty} J_r^\lambda = 0$ without difficulty. Next consider the smallest α such that $D_\alpha \subset \Delta + \sum_{\beta \leq \delta} P_\beta$ and D_α has not yet been dealt with in any of the previous steps. We then construct by our lemma countably many closed Jordan curves J_r^δ such that $D_\alpha \cap J_r^\delta$ is dense on J_r^δ, $J_r^\delta \cap (\Delta + \sum_{\beta<\delta} P_\beta - D_\alpha) = 0$, and any two points p and q can be separated by some J_r^δ. This completes the δth step. Thus our construction can be carried out through all ordinals of the second number class and will exhaust all the C_α's and D_α's. Consider now the set $A = \Delta + \sum_\beta P_\beta$. We claim that no finite set disconnects it and that any countable dense set totally disconnects it. First of all every cut of the plane intersects our set in an infinite set, thus no finite set can disconnect it. On the other hand let D_α be some countable dense set of A (clearly A is dense in the plane). At some step in our construction we had to deal with D_α and it is clear that $\Delta + \sum_\beta P_\beta - D_\alpha$ is totally disconnected, since if p and q are any two points there exists a J_r separating them and $(\Delta + \sum_{\beta<\alpha} P_\beta - D_\alpha) \cap J_r = 0$. q.e.d.

By a slight modification of Miller's[8] construction of a biconnected set without a dispersion point, we can construct a connected set A dense in an indecomposable continuum, such that there exists an open set O of the plane with $O \cap A$ having power c and if B is any connected subset of A then $A - B$ is nowhere dense in $O \cap A$. Clearly such a set is biconnected and in fact it can be made to have no dispersion point. By a further slight modification of Miller's construction we can construct a biconnected set without a dispersion point such that if $A_1, A_2, \cdots$ is any countable collection of connected subsets of A, then $\bigcup_i (A - A_i) \neq A$ or $\bigcap_i A_i \neq 0$.

I can not decide the question whether there exists a connected set such that the complement of every connected subset of it is nowhere dense in it.

The following problems may be of some interest: Is it true that every connected set contains a connected subset not homeomorphic to it? (Points do not count as connected sets.) Every known connected set (which is subset of a Euclidean space) contains at least three different types of connected subsets. The number three can not be improved as is shown by the arc.

If a topological space does not satisfy any separation axioms the following example communicated to me by S. Eilenberg shows that the above conjecture is not true: Let the space be the integers, the

[8] Miller, ibid.

closed sets are the finite sets. Every infinite set is connected and they are clearly all homeomorphic.

Is it true that every connected set of dimension n contains a connected subset of dimension $n-1$?

Is it true that every connected set of dimension greater than 1 contains 2^c connected subsets?

Perhaps the following theorem which A. Stone and I proved might be of some interest: Let A be a totally disconnected set of power m (in a separable space), then if m is not the sum of countably many smaller cardinals, A can be written as the sum of two separated sets of power m. If m is the sum of $\aleph_0$ smaller cardinals the theorem is in general false.

PROOF. Suppose first $m \neq \sum_{k=1}^{\infty} m_k$, $m_k < m$. Denote by A' the set of those points of A for which every neighborhood contains m points of A. Now $A-A'$ has power less than m, since by Lindelöf's theorem $A-A'$ can be covered by countably many open sets each of which contains less than m points, thus $A-A'$ also contains less than m points. Since A is totally disconnected $A'=U+V$, where U and V are open-closed sets. Thus both of them have power m. Let $p \in A-A'$, the distance of p from U be $f(p)$, from V, $\phi(p)$, $p \in U_\alpha$ if $f(p) \leqq \alpha\phi(p)$, $p \in V_\alpha$ if $f(p) \geqq \alpha\phi(p)$. Since the power of $A-A'$ is less than $m \leqq c$ there exists an α such that $U_\alpha \cap V_\alpha = 0$. But then $U+U_\alpha$ and $V+V_\alpha$ are two separated sets of power m whose sum is A; this completes the proof.

If $m = \sum_{k=1}^{\infty} m_k$, $m_k < m$, we define A as follows: A contains m_k points in $(1/k, 1/k+1)$, 0 belongs to A. Clearly A is totally disconnected (its power is less than c, since c is not the sum of m smaller cardinals). Obviously A is not the sum of two separated sets of power m.

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