ALMOST ORTHOGONAL SERIES

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1. Almost orthogonal series. Let us consider an infinite sequence 
\{\phi_n(x)\}, n = 1, 2, \ldots, of complex-valued functions of the real variable x, of class \(L^2(a, b)\), normalized so that \(\int_a^b |\phi_n(x)|^2 dx = 1\) for all n. Assume further that the sequence satisfies the following condition

\[
\sum_{m,n} |a_{mn}|^2 < \infty,
\]

where \(a_{mn} = \int_a^b \phi_m \overline{\phi_n} dx\) (\(m \neq n; n, m = 1, 2, \ldots\)), \(a_{mn} = 0, m = n\).

We wish to show that under the above conditions we have a Bessel inequality and an analogue of the Riesz-Fisher theorem.

**Theorem 1 (Bessel’s inequality).** Under the above conditions, let \(f(x)\) be a real-valued function belonging to \(L^1(a, b)\), and \(b_n = \int_a^b \phi_n dx\), then

\[
\sum_{1}^{\infty} |b_n|^2 \leq \int_a^b |f|^2 dx \left[ 1 + \left( \sum_{m,n} |a_{mn}|^2 \right)^{1/2} \right].
\]

We have

\[
\sum_{1}^{n} |b_n|^2 = \int_a^b f \left[ \sum_{1}^{n} b_n \overline{\phi_n} \right] dx.
\]

Using Schwartz’s inequality, this becomes

\[
\sum_{1}^{n} |b_n|^2 \leq \left[ \int_a^b |f|^2 dx \right]^{1/2} \left[ \int_a^b \left[ \sum_{1}^{n} b_n \phi_n \right] \left[ \sum_{1}^{n} b_n \overline{\phi_n} \right] dx \right]^{1/2}
\]

\[
\leq \left[ \int_a^b |f|^2 dx \right]^{1/2} \left[ \sum_{1}^{n} |b_n|^2 + \sum_{1,1, k \neq l}^{n,n} b_k b_l \overline{a_{kl}} \right]^{1/2}
\]

\[
\leq \left[ \int_a^b |f|^2 dx \right]^{1/2} \left[ \sum_{1}^{n} |b_n|^2 \right]^{1/2} + \left\{ \sum_{1,1}^{n,n} |b_k|^2 \right\}^{1/2} \left\{ \sum_{1,1}^{n,n} |a_{kl}|^2 \right\}^{1/2} \left[ 1 + \left( \sum_{m,n} |a_{mn}|^2 \right)^{1/2} \right]^{1/2}
\]

\[
\leq \left[ \int_a^b |f|^2 dx \right]^{1/2} \left[ \sum_{1}^{n} |b_n|^2 \right]^{1/2} \cdot \left[ 1 + \left( \sum_{m,n} |a_{mn}|^2 \right)^{1/2} \right].
\]

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Hence, simplifying,
\[ \sum_{1}^{n} |b_k|^2 \leq \left( \int_{a}^{b} |f|^2 \, dx \right) \left[ 1 + \left( \sum_{m,n} |a_{kl}|^2 \right)^{1/2} \right], \]
and since this is true for all \( n \), the result is obtained.

**Theorem 2 (Analogue of Riesz-Fisher).** Under the above conditions, if \( \sum_{n} |b_n|^2 < \infty \), there exists a function \( f(x) \subset L^2(a, b) \) such that
\[ \sum_{n} \left| b_k - \int_{a}^{b} f^k \phi d^k x \right|^2 \leq \left( \sum_{n} |b_k|^2 \right) \left( \sum_{k,l} |a_{kl}|^2 \right), \]
and therefore \( \lim_{n \to \infty} (b_n - \int_{a}^{b} f \phi d^k x) = 0. \)

Let \( s_n = \sum_{k}^n b_k \phi_k \), then
\[
\begin{align*}
\int_{a}^{b} |s_m - s_n|^2 \, dx &= \int_{a}^{b} \left[ \sum_{n+1}^{m} b_k \phi_k \right] \left[ \sum_{n+1}^{m} b_k \phi_k \right] \, dx \\
&= \sum_{n+1}^{m} |b_k|^2 + \sum_{n+1}^{m} b_k b_{l} a_{kl} \\
&\leq \sum_{n+1}^{m} |b_k|^2 + \left( \sum_{n+1}^{m} |b_k|^2 |b_l|^2 \right)^{1/2} \left( \sum_{m,n} |a_{mn}|^2 \right)^{1/2} \\
&\leq \sum_{n+1}^{m} |b_k|^2 \left( 1 + \left( \sum_{m,n} |a_{mn}|^2 \right)^{1/2} \right)
\end{align*}
\]
(using Schwartz's inequality). Hence, since \( \sum b_n^2 < \infty \), \( s_n \) converges in mean to a function \( f(x) \subset L^2(a, b) \), that is,
\[ \lim_{n \to \infty} \int_{a}^{b} |f(x) - s_n(x)|^2 \, dx = 0. \]

We have, \( k < n, \)
\[
\begin{align*}
b_k - \int_{a}^{b} f \phi_k \, dx &= b_k - \int_{a}^{b} s_n \phi_k \, dx + \int_{a}^{b} (s_n - f) \phi_k \, dx, \\
\int_{a}^{b} |s_n - f| \phi_k \, dx &\leq \left( \int_{a}^{b} |f - s_n|^2 \, dx \right)^{1/2} \left( \int_{a}^{b} |\phi_k|^2 \, dx \right)^{1/2} \\
&= \left( \int_{a}^{b} |f - s_n|^2 \, dx \right)^{1/2},
\end{align*}
\]
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\[ b_k - \int_a^b s_n \bar{\phi}_k dx = \sum_{j=1}^{n} b_j a_{kj}, \]

\[ \left| b_k - \int_a^b s_n \bar{\phi}_k dx \right| \leq \sum_{j=1}^{n} |b_j a_{kj}| \]

\[ \leq \left( \sum_{j=1}^{n} |b_j|^2 \right)^{1/2} \left( \sum_{j=1}^{n} |a_{kj}|^2 \right)^{1/2}. \]

Hence,

\[ \left| b_k - \int_a^b f \bar{\phi}_k dx \right| \leq \left( \sum_{j=1}^{n} |b_j|^2 \right)^{1/2} \left( \sum_{j=1}^{n} |a_{kj}|^2 \right)^{1/2} \]

\[ + \left( \int_a^b |f - s_n|^2 dx \right)^{1/2}. \]

Let \( n \to \infty \), this becomes

\[ \left| b_k - \int_a^b f \bar{\phi}_k dx \right| \leq \left( \sum_{j=1}^{\infty} |b_j|^2 \right)^{1/2} \left( \sum_{j=1}^{\infty} |a_{kj}|^2 \right)^{1/2}. \]

Squaring both sides, and summing over \( k \),

\[ \sum_k \left| b_k - \int_a^b f \bar{\phi}_k dx \right|^2 \leq \left( \sum_{j=1}^{\infty} |b_j|^2 \right) \left( \sum_{k,j} |a_{kj}|^2 \right), \]

which is the result in question.

2. An "almost" moment problem. Let us consider the sequence of functions \( \{e^{\lambda_n t}/(b-a)^{1/2}\} \), the \( \lambda_n \) being real and distinct, over a finite interval \((a, b)\). Then we have the following theorem.

**Theorem 3.** If \( \sum_{k=1}^{\infty} (\lambda_k - \lambda_l)^2 < \infty \), and \( \sum_n |b_n|^2 < \infty \), there exists a function \( f(t) \in L^2(a, b) \), such that

\[ \sum_n \left| b_n - \frac{1}{(b-a)^{1/2}} \int_a^b f(t)e^{-\lambda_n t} dt \right|^2 < \infty. \]

We have

\[ \left| \int_a^b e^{\lambda_k t} e^{-\lambda_l t} dt \right| \leq \frac{2}{|\lambda_k - \lambda_l|}, \quad k \neq l. \]

Therefore, in view of the hypothesis, this is a corollary of Theorem 2.

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