ON CERTAIN ARITHMETICAL FUNCTIONS DUE TO G. HUMBERT

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1. **Introduction.** G. Humbert has discussed, in a series of brief notes,[1] a certain class of entire functions with interesting arithmetical properties. These functions are defined, in an essentially unique manner, by certain functional equations. The Fourier series representations of the solutions of these equations are similar in form to those for the elliptic functions *snu*, *cnu*, *dnu*, and so on. They differ from these, however, in that their domain of validity extends throughout the entire complex plane (Z_∞ excluded) and moreover, in that their arithmetized forms involve *incomplete* numerical functions of the divisors of an integer.

In the present paper we extend somewhat the results of Humbert and obtain a relation between his functions and certain pseudo-periodic functions discussed elsewhere by the writer.[2] This relation is, in effect, embodied in a series of twelve identities; these are of some interest in that their arithmetical equivalents (paraphrases) are relatively simple and involve partitions related to the representations of an integer as the sum of five squares.

It is also pointed out that as an immediate consequence of the analytical form of Humbert's functions, it is possible to deduce a series of relations between the greatest integer function $E(x)$ and certain incomplete numerical functions of the divisors of an integer.

2. **The functional equations.** In what follows the notation is that ordinarily used in the theory of the elliptic theta functions.[3] The period $\pi\tau$ is such that $0 < \arg \tau < \pi$.

The set of functional equations considered has the form

$$\text{(A)} \qquad \begin{aligned} h(z+\pi) &= (-1)^a h(z), \\ h(z+\pi\tau) &= (-1)^b h(z) + F_{ab}^{(\alpha)}(z), \end{aligned}$$

where a, b take the values zero or unity, and $F_{ab}^{(\alpha)}(z)$, to be defined presently, is an expression which involves the theta function $\vartheta_\alpha(z)$.

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[1] G. Humbert, *Sur quelques fonctions numeriques remarquables*, C. R. Acad. Sci. Paris vol. 158 (1914) pp. 220, 294, and 1841; vol. 163 (1916) p. 412.

[2] M. A. Basoco, Amer. J. Math. vol. 54 (1932) pp. 242–252.

[3] Whittaker and Watson, *Modern analysis*, Cambridge.

We shall denote the integral functions satisfying these equations by the symbol $H_{ab}^{(\alpha)}(z)$. These are readily found on assuming series solutions of the form

$$H_{ab}^{(\alpha)}(z) = \sum_{k=-\infty}^{\infty} A_k e^{ikz}.$$

These solutions, for $(a, b) = (1, 0)$, $(0, 1)$ and $(1, 1)$, are unique; for $(a, b) = (0, 0)$, the solution is completely determined to within an additive constant. For suppose that for a given (a, b) there exist two distinct solutions. Denote by $D(z)$ their difference; this would likewise be an integral function and would satisfy periodicity relations of the form

$$D(z + \pi) = (-1)^a D(z),$$
$$D(z + \pi\tau) = (-1)^b D(z).$$

The function $D(z)$ is therefore an elliptic function, reducing by Liouville's theorem to a constant. From (A) it follows easily that this constant vanishes except for the case $(a, b) = (0, 0)$, when it remains undetermined. In this case, we have selected the solution which vanishes for $z = 0$.

3. **The functions** $F_{ab}^{(\alpha)}(z)$. Let λ and μ be the multipliers associated with the reduction of the theta functions of argument $z + \pi\tau/2$ and $z + \pi\tau$ respectively, so that

$$\lambda = q^{-1/4} e^{-iz}, \qquad \mu = q^{-1} e^{-2iz}$$

where

$$q = \exp \pi i \tau, \qquad 0 < \arg \tau < \pi.$$

The functions $F_{ab}^{(\alpha)}(z)$ are defined as follows:

$$F_{00}^{(0)}(z) = i(1 - \mu)\vartheta_0(z) - 2i; \qquad F_{11}^{(0)}(z) = 2i\lambda\vartheta_0(z),$$
$$F_{00}^{(1)}(z) = 2i(-1 + i\lambda\vartheta_1(z)); \qquad F_{11}^{(1)}(z) = i(1 + \mu)\vartheta_1(z),$$
$$F_{00}^{(2)}(z) = 2i(-1 + \lambda\vartheta_2(z)); \qquad F_{11}^{(2)}(z) = i(1 - \mu)\vartheta_2(z),$$
$$F_{00}^{(3)}(z) = i(1 + \mu)\vartheta_3(z) - 2i; \qquad F_{11}^{(3)}(z) = 2\lambda\vartheta_3(z),$$

. . . .

$$F_{01}^{(0)}(z) = i(1 + \mu)\vartheta_0(z); \qquad F_{10}^{(0)}(z) = 2\lambda\vartheta_0(z),$$
$$F_{01}^{(1)}(z) = 2i\lambda\vartheta_1(z); \qquad F_{10}^{(1)}(z) = -i(1 - \mu)\vartheta_1(z),$$
$$F_{01}^{(2)}(z) = 2\lambda\vartheta_2(z); \qquad F_{10}^{(2)}(z) = i(1 + \mu)\vartheta_2(z),$$
$$F_{01}^{(3)}(z) = i(1 - \mu)\vartheta_3(z); \qquad F_{10}^{(3)}(z) = 2i\vartheta_3(z).$$

These functions satisfy the condition

$$F_{ab}^{(\alpha)}(z+\pi) = (-1)^a F_{ab}^{(\alpha)}(z),$$

which is implied by equations (A).

4. **The solutions** $H_{ab}^{(\alpha)}(z)$. The procedure indicated in §2 yields the solutions $H_{ab}^{(\alpha)}(z)$ of equations (A) corresponding to the choices of $F_{ab}^{(\alpha)}(z)$ listed in §3. These solutions are valid for all values of z. We thus obtain:

$$H_{00}^{(3)}(z) = 2\sum_{(n)} \frac{q^{n^2+2n} + q^{n^2}}{1-q^{2n}} \sin 2nz, \tag{4.1}$$

$$H_{00}^{(2)}(z) = 4\sum_{(n)} \frac{q^{n^2+n}}{1-q^{2n}} \sin 2nz, \tag{4.2}$$

$$H_{01}^{(3)}(z) = 2\sum_{(n)} \frac{q^{n^2+2n} - q^{n^2}}{1+q^{2n}} \sin 2nz, \tag{4.3}$$

$$H_{01}^{(2)}(z) = 1 + 4\sum_{(n)} \frac{q^{n^2+n}}{1+q^{2n}} \cos 2nz, \tag{4.4}$$

$$H_{10}^{(3)}(z) = 4\sum_{(m)} \frac{q^{(m^2+2m)/4}}{1-q^{m}} \sin mz, \tag{4.5}$$

$$H_{10}^{(2)}(z) = 2\sum_{(m)} \frac{q^{(m^2+4m)/4} + q^{m^2/4}}{1-q^{m}} \sin mz, \tag{4.6}$$

$$H_{11}^{(3)}(z) = 4\sum_{(m)} \frac{q^{(m^2+2m)/4}}{1+q^{m}} \cos mz, \tag{4.7}$$

$$H_{11}^{(2)}(z) = 2\sum_{(m)} \frac{q^{(m^2+4m)/4} - q^{m^2/4}}{1+q^{m}} \sin mz. \tag{4.8}$$

In the preceding the index of summation n ranges over all the positive integers while the index m ranges over the odd positive integers.

The remaining functions $H_{ab}^{(c)}(z)$ $(c=0, 1)$ may be obtained from the above results upon replacing z by $z+\pi/2$.

5. **Arithmetized form.** In this section we list the arithmetical forms of the trigonometric series (4.1) to (4.8). The following notation is used: n denotes an arbitrary positive integer; m is an arbitrary positive *odd* integer; α is a positive integer of the form $4k+1$ and β is one of the form $4k+3$; $\sum'$ refers to the conjugate divisors (d, δ) of n and (t, τ) of α or β such that $\delta<d$, $\tau<t$; $\epsilon(n)$ is one or zero according as n is or is not the square of an integer.

$$(5.1)\quad H_{00}^{(3)}(z) = 2\sum_{(n)} q^n \epsilon(n) \sin 2n^{1/2} z + 4\sum_{(n)} q^n \left\{ \sum{}' \sin 2\delta z \right\}, \qquad (\delta - d \equiv 0, \bmod 2).$$

$$(5.2)\quad H_{00}^{(2)}(z) = 4\sum_{(n)} q^n \left\{ \sum{}' \sin 2\delta z \right\}, \qquad (\delta - d \equiv 1, \bmod 2).$$

$$(5.3)\quad \begin{aligned} H_{01}^{(3)}(z) &= -2\sum_{(n)} q^n \epsilon(n) \sin 2n^{1/2} z \\ &\quad - 4\sum_{(n)} q^n \left\{ \sum{}'(-1)^{(d-\delta)/2} \sin 2\delta z \right\}, \qquad (\delta - d \equiv 0, \bmod 2). \end{aligned}$$

$$(5.4)\quad H_{01}^{(2)}(z) = 1 + 4\sum_{(n)} q^n \left\{ \sum{}'(-1)^{(\delta-d-1)/2} \cos 2\delta z \right\}, \qquad (\delta - d \equiv 1, \bmod 2).$$

$$(5.5)\quad H_{10}^{(3)}(z) = 4\sum_{(\beta)} q^{\beta/4} \left\{ \sum{}' \sin \tau z \right\},$$

$$(5.6)\quad H_{10}^{(2)}(z) = 2\sum_{(\alpha)} q^{\alpha/4} \epsilon(\alpha) \sin \alpha^{1/2} z + 4\sum q^{\alpha/4} \left\{ \sum{}' \sin \tau z \right\},$$

$$(5.7)\quad H_{11}^{(3)}(z) = 4\sum_{(\beta)} q^{\beta/4} \left\{ \sum{}'(-1)^{(t-\tau-2)/4} \cos \tau z \right\},$$

$$(5.8)\quad \begin{aligned} H_{11}^{(2)}(z) &= -2\sum_{(\alpha)} q^{\alpha/4} \epsilon(\alpha) \sin \alpha^{1/2} z \\ &\quad - 4\sum_{(\alpha)} q^{\alpha/4} \left\{ \sum{}'(-1)^{(t-\tau)/4} \sin \tau z \right\}. \end{aligned}$$

The remaining eight functions will not be listed explicitly; they can be obtained quite readily from what precedes.

6. **Identities.** In a former paper[4] the writer obtained the trigonometric developments for the sixteen theta quotients of the form

$$\Theta_{\alpha\beta\gamma}(x, y) \equiv \vartheta_1'^2 \frac{\vartheta_\alpha(x + y)}{\vartheta_\beta^2(x)\vartheta_\gamma(y)} \cdot$$

A comparison of these expansions (with $(x, y) = (0, 2)$ and $\beta \neq 1$) with those given in the preceding sections leads to the following identities:

$$(6.1)\quad \begin{aligned} \vartheta_1'(z) H_{01}^{(1)}(z) &= \vartheta_0^2\vartheta_3^2\vartheta_2(z) \\ &\quad - 4\vartheta_1(z) \sum_{(n)}{}' q^n \left\{ \sum{}'(-1)^{(d+\delta-1)/2} (d + \delta) \sin 2\delta z \right\} \\ &\qquad (\delta - d \equiv 1, \bmod 2), \end{aligned}$$

$$(6.2)\quad \begin{aligned} \vartheta_2'(z) H_{01}^{(2)}(z) &= -\vartheta_0^2\vartheta_3^2\vartheta_1(z) \\ &\quad - 4\vartheta_2(z) \sum_{(n)} q^n \left\{ \sum{}'(-1)^{(d-\delta-1)/2} (d + \delta) \sin 2\delta z \right\} \\ &\qquad (\delta - d \equiv 1, \bmod 2), \end{aligned}$$

[4] Loc. cit.

$$(6.3)\quad \vartheta_3'(z)H_{10}^{(3)}(z) = -\vartheta_2^2\vartheta_3^2\vartheta_3(z) + 2\vartheta_3(z)\sum_{(\beta)} q^{\beta/4}\left\{\sum{}'(t+\tau)\cos\tau z\right\},$$

$$(6.4)\quad \begin{aligned}\vartheta_3'(z)H_{11}^{(3)}(z) &= \vartheta_0^2\vartheta_2^2\vartheta_1(z)\\ &\quad - 2\vartheta_3(z)\sum_{(\beta)} q^{\beta/4}\left\{\sum{}'(-1)^{(t-\tau-2)/4}(t+\tau)\sin\tau z\right\},\end{aligned}$$

$$(6.5)\quad \begin{aligned}\vartheta_0'(z)H_{11}^{(0)}(z) &= -\vartheta_0^2\vartheta_2^2\vartheta_2(z)\\ &\quad - 2\vartheta_3(z)\sum_{(\beta)} q^{\beta/4}\left\{\sum{}'(-1)^{(t+\tau)/4}(t+\tau)\cos\tau z\right\},\end{aligned}$$

$$(6.6)\quad \begin{aligned}\vartheta_0'(z)H_{10}^{(0)}(z) &= \vartheta_2^2\vartheta_3^2\vartheta_1(z)\\ &\quad - 2\vartheta_0(z)\sum_{(\beta)} q^{\beta/4}\left\{\sum{}'(-1)^{(\tau-1)/2}(t+\tau)\sin\tau z\right\},\end{aligned}$$

$$(6.7)\quad \begin{aligned}\vartheta_1'(z)H_{10}^{(1)}(z) &= \vartheta_2^2\vartheta_3^2\vartheta_0(z)\\ &\quad - 2\vartheta_1(z)\sum_{(\alpha)} q^{\alpha/4}\left\{(-1)^{(\alpha^{1/2}-1)/2}\epsilon(\alpha)\alpha^{1/2}\sin\alpha^{1/2}z\right.\\ &\qquad\left. + \sum{}'(-1)^{(\tau-1)/2}(t+\tau)\sin\tau z\right\},\end{aligned}$$

$$(6.8)\quad \begin{aligned}\vartheta_1'(z)H_{11}^{(1)}(z) &= \vartheta_0^2\vartheta_2^2\vartheta_3(z)\\ &\quad - 2\vartheta_1(z)\sum_{(\alpha)} q^{\alpha/4}\left\{(-1)^{(\alpha^{1/2}-1)/2}\epsilon(\alpha)\alpha^{1/2}\sin\alpha^{1/2}z\right.\\ &\qquad\left. + \sum{}'(-1)^{(t+\tau-2)/4}(t+\tau)\sin\tau z\right\},\end{aligned}$$

$$(6.9)\quad \begin{aligned}\vartheta_2'(z)H_{10}^{(2)}(z) &= -\vartheta_2^2\vartheta_3^2\vartheta_3(z) + 2\vartheta_2(z)\sum_{(\alpha)} q^{\alpha/4}\left\{\epsilon(\alpha)\alpha^{1/2}\cos\alpha^{1/2}z\right.\\ &\qquad\left. + \sum{}'(t+\tau)\cos\tau z\right\},\end{aligned}$$

$$(6.10)\quad \begin{aligned}\vartheta_2'(z)H_{11}^{(2)}(z) &= -\vartheta_0^2\vartheta_2^2\vartheta_0(z) + 2\vartheta_2(z)\sum_{(\alpha)} q^{\alpha/4}\left\{\epsilon(\alpha)\alpha^{1/2}\cos\alpha^{1/2}z\right.\\ &\qquad\left. + \sum{}'(-1)^{(t-\tau)/4}(t+\tau)\cos\tau z\right\},\end{aligned}$$

$$(6.11)\quad \begin{aligned}\vartheta_0'(z)H_{01}^{(0)}(z) &= \vartheta_0^2\vartheta_3^2\vartheta_3(z) - \vartheta_0(z)\left\{\psi(z)\right.\\ &\qquad\left. - 4\sum_{(n)} q^n\left\{\sum{}'(-1)^{(d+\delta)/2}(d+\delta)\cos 2\delta z\right\}\right\}\\ &\qquad (\delta - d \equiv 0, \bmod 2),\end{aligned}$$

$$(6.12)\quad \begin{aligned}\vartheta_3'(z)H_{01}^{(3)}(z) &= -\vartheta_0^2\vartheta_3^2\vartheta_0(z) + \vartheta_3(z)\left\{\phi(z)\right.\\ &\qquad\left. - 4\sum_{(n)} q^n\left\{\sum{}'(-1)^{(\delta-d)/2}(d+\delta)\cos 2\delta z\right\}\right\}\\ &\qquad (\delta - d \equiv 0, \bmod 2),\end{aligned}$$

where

$$\psi(z) = 1 - 4\sum_{n=1}^{\infty}(-1)^n n q^{n^2}\cos 2nz$$

and

$$\phi(z) = \psi(z + \pi/2).$$

The above yield, therefore, identities for the product $\vartheta_\alpha'(z)H_{ab}^{(\alpha)}(z)$, the case $(a, b) = (0, 0)$ being, however, excluded. This is because in the expansions for the functions $\Theta_{\alpha\beta\gamma}(x, y)$ with $\beta = 1$ it is not possible to set $x = 0$, this point being a pole of these functions.

7. **Paraphrases.** The preceding set of identities may be paraphrased[5] into rather simple arithmetical equivalents. It is of interest to note that the partitions involved in these paraphrases refer to the representations of numbers in certain linear forms as the sum of five squares.

The notation used is as follows: $\alpha, \beta, m, m_1, n, n_1, t, \tau, d, \delta$ are positive integers; $\alpha \equiv 1$ and $\beta \equiv 3 \pmod 4$ while m, m_1, τ are odd; n and n_1 are unrestricted. The sets of conjugate divisors (d, δ) and (t, τ) are subject to the condition $\delta < d$ and $\tau < t$; further restrictions on these divisors will be indicated as needed. h, z_i are unrestricted integers, positive, negative, or zero, while μ, $n_i \gtrless 0$ are odd; the $w_i \leqq 0$, w_i are even integers, positive, negative, or zero. Moreover, $\epsilon(n) = 1$ or 0 according as n is or is not the square of an integer and $\tau(n) = 1$ or 0 accordingly as n is or is not the sum of two integral squares. Finally, the functions $f(x)$ and $g(x)$ are quite arbitrary except that they must be respectively even and odd and be well defined for integral values of the argument.

The partitions of the integers $\alpha, \beta, 2m, n$ which appear in our results are as follows:

$$(i) \qquad \alpha = x_1^2 + w_1^2 + w_2^2 + w_3^2 + w_4^2 = \mu^2 + 4d\delta \qquad (\delta - d \equiv 1, \bmod 2),$$

$$(j) \qquad \beta = x_1^2 + x_2^2 + x_3^2 + w_1^2 + w_2^2 = 4h^2 + t\tau,$$

$$(k) \qquad 2m = x_1^2 + x_2^2 + w_1^2 + w_2^2 + w_3^2 = \mu^2 + t\tau = \mu^2 + m_1^2,$$

$$(l) \qquad n = z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = h^2 + d\delta = h^2 + n_1^2 \qquad (\delta - d \equiv 0, \bmod 2).$$

The arithmetical equivalents of (6.1) to (6.12) follow in the same order; the necessary partition is denoted by placing the proper letter i, j, k or l under the first $\sum$. In the relations (7.1) and (7.2) the divisors (d, δ) are of opposite parity while in (7.11) and (7.12) they have like parity.

[5] E. T. Bell, Trans. Amer. Math. Soc. vol. 22 (1921) pp. 1–39 and 198–219.

$$\sum_{(i)} (-1)^{(w_1+w_2)/2} f(x_1) = 2\epsilon(\alpha)\alpha^{1/2}(-1)^{(\alpha^{1/2}-1)/2} f(\alpha^{1/2}) \tag{7.1}$$
$$+4\sum{}'(-1)^{(d+\delta+\mu)/2}(d+\delta-\mu)f(2\delta+\mu),$$

$$\sum_{(i)} (-1)^{(w_1+w_2+x_1-1)/2} g(x_1) = 2\epsilon(\alpha)\alpha^{1/2} g(\alpha^{1/2}) \tag{7.2}$$
$$+4\sum{}'(-1)^{(\delta-d+1)/2}(d+\delta-\mu)g(2\delta+\mu),$$

$$\sum_{(j)} f(x_1) = 2\sum{}'(t+\tau-4h)f(\tau+2h), \tag{7.3}$$

$$\sum_{(j)} (-1)^{(w_1+w_2+x_1-1)/2} g(x_1) = 2\sum{}'(-1)^{(t-\tau-2)/4}(t+\tau-4h)g(\tau+2h), \tag{7.4}$$

$$\sum_{(j)} (-1)^{(w_1+w_2)/2} f(x_1) = -2\sum{}'(-1)^{(t+\tau+4h)/2}(t+\tau-4h)f(\tau+2h), \tag{7.5}$$

$$\sum_{(j)} (-1)^{(x_1-1)/2} g(x_1) = 2\sum{}'(-1)^{(\tau+2h-1)/2}(t+\tau-4h)g(\tau+2h), \tag{7.6}$$

$$\sum_{(k)} (-1)^{w_1/2} f(w_1) = 2\sum{}'(-1)^{(\tau+\mu)/2}(t+\tau-2\mu)f(\tau+\mu) \tag{7.7}$$
$$+2\lambda(2m)\sum (-1)^{(m_1+\mu)/2}(m_1-\mu)f(m_1+\mu),$$

$$\sum_{(k)} (-1)^{(w_1+w_2)/2} f(w_3) = 2\sum{}'(-1)^{(t+\tau+2\mu)/4}(t+\tau-2\mu)f(\tau+\mu) \tag{7.8}$$
$$+2\lambda(2m)\sum (-1)^{(m_1+\mu)/2}(m_1-\mu)f(m_1+\mu),$$

$$\sum_{(k)} f(w_1) = 2\sum{}'(t+\tau-2\mu)f(\tau+\mu) + 2\lambda(2m)\sum (m_1-\mu)f(m_1+\mu), \tag{7.9}$$

$$\sum_{(k)} (-1)^{(w_1+w_2+w_3)/2} f(w_1) = 2\sum{}'(-1)^{(t-\tau)/4}(t+\tau-2\mu)f(\tau+\mu) \tag{7.10}$$
$$+2\lambda(2m)\sum (m_1-\mu)f(m_1+\mu),$$

$$\sum_{(l)} (-1)^{z_1+z_2} f(z_3) = f(0) + 2\epsilon(n)(-1)^{n^{1/2}} f(n^{1/2}) \tag{7.11}$$
$$+4\lambda(n)\sum (-1)^{h+n_1}(h-n_1)f(h+n_1)$$
$$-4\sum{}'(-1)^{(d+\delta+2h)/2}(d+\delta-2h)f(\delta+h),$$

$$\sum_{(l)} (-1)^{z_1+z_2+z_3} f(z_1) = f(0) + 2\epsilon(n) f(n^{1/2}) \tag{7.12}$$
$$+4\lambda(n)\sum (h-n_1)f(h+n_1)$$
$$-4\sum{}'(-1)^{(d-\delta)/2}(d+\delta-2h)f(\delta+h).$$

The preceding formulae with $f(x)=1$ yield enumerations relative to the number of representations of a number as the sum of five squares. The most interesting results are those deduced from (7.3) and (7.9). We thus obtain the following theorems.

THEOREM A. *If* $\beta\equiv 3$, mod 4, *and* $x, y, z\gtrless 0$ *are odd while* u, v *are even integers, positive, negative, or zero, then*

$$N[\beta = x^2 + y^2 + z^2 + u^2 + v^2] = 2\phi(\beta) + 4\sum_{(r)} \phi(\beta - 4r^2),$$

where $r=1, 2, 3, \cdots, [\beta^{1/2}/2]$, *and* $\phi(n)$ *is the sum of the positive integral divisors of* n.

THEOREM B. *If* $m\equiv 1$, mod 2, *and* $x, y\gtrless 0$ *are odd while* u, v, w *are even and positive, negative, or zero, then*

$$N[2m = x^2 + y^2 + u^2 + v^2 + w^2] = 4\sum_{(s)} \phi(2m - s^2),$$

where $s=1, 3, 5, 7, \cdots, [(2m)^{1/2}]$, *provided* $2m$ *is not representable as the sum of two squares. If, however,* $2m$ *is so representable, the quantity* $G(2m)$ *must be added to the preceding sum, where* $G(2m)=4\sum x$, *the sum being extended over all solutions of* $2m=x^2+y^2$, $x, y>0$ *and odd.* $\phi(n)$ *is as in the preceding theorem.*

8. **Application to the function** $E(x)$. The analytical form of the functions defined by (4.1) to (4.8) suggests the application of a device due to Hermite,[6] which yields identities involving the greatest integer function $E(x)$. Hermite's method depends on the following generating functions:

$$\frac{u^b}{(1-u)(1-u^a)} = \sum_{(n)} E\left(\frac{n+a-b}{a}\right)u^n;$$

$$\frac{u^b}{(1-u)(1+u^a)} = \sum_{(n)} E_1\left(\frac{n+a-b}{2a}\right)u^n,$$

where a, b are positive integers and

$$E_1(x) = E(2x) - 2E(x) = E(x+1/2) - E(x).$$

The following two relations are typical of the set of sixteen which may be deduced from our results. Let $F(z)$ be an arbitrary function, r a positive integer and (d, δ) conjugate integral divisors of r. Define $P_1(x, r)$ and $P_2(x, r)$ by the following:

$$P_1(x, r) = \epsilon(r)F(2r^{1/2}x) + 2\sum{}' F(2\delta n) \quad (r = d\delta,\ d-\delta\equiv 0, \text{mod } 2),$$

$$P_2(x, r) = \sum{}' F(2\delta x) \quad (r = d\delta,\ d-\delta\equiv 1, \text{mod } 2).$$

[6] Hermite, Acta Math. vol. 5 (1884–1885) pp. 297–330; J. Reine Angew. Math. vol. 100 (1887) pp. 51–65; *Oeuvres*, vol. 5, pp. 151–159. See also a paper by the present writer in Bull. Amer. Math. Soc. vol. 42 (1936) pp. 720–726.

Let n be an arbitrary fixed positive integer; then

$$(8.1)\qquad \sum_{r=1}^{n} P_1(x, r) = 2\sum_{(s)} E\left(\frac{n - s^2}{2s}\right)F(2sx) + \sum_{s=1}^{[n^{1/2}]} F(2sx),$$

$$(8.2)\qquad \sum_{r=1}^{n} P_2(x, r) = \sum_{(s)} E\left(\frac{n + s - s^2}{2s}\right)F(2sx),$$

where s ranges over the values $s=1, 2, 3, \cdots$ so long as the argument in $E(x)$ is greater than or equal to 1.

In particular, let $F(z)=z^k$, where k is a positive integer. Then the preceding reduce to the following:

$$(8.3)\qquad \sum_{r=1}^{n} X_1(r) = 2\sum_{(s)} E\left(\frac{n - s^2}{s}\right)s^k + \sum_{s=1}^{[n^{1/2}]} s^k,$$

$$(8.4)\qquad \sum_{r=1}^{n} X_2(r) = \sum_{(s)} E\left(\frac{n + s - s^2}{2s}\right)s^k,$$

where,

$$X_1(r) = \epsilon(r)r^{1/2} + 2\sum{}'\delta^k \qquad (r = d\delta,\ \delta < d,\ \delta - d \equiv 0, \bmod 2),$$

$$X_2(r) = \sum{}'\delta^k \qquad (r = d\delta,\ \delta < d,\ \delta - d \equiv 1, \bmod 2).$$

These results follow from (4.1), (4.2) and their equivalents (5.1) and (5.2) respectively.

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