ON A THEOREM OF BOHR AND PÁL

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Let D be the domain bounded by a simple closed plane Jordan curve of equations $x=f(t)$, $y=g(t)$, where f and g are continuous and of period 2π. Fejér [1][1] has proved that the power series representing the function mapping conformally the interior of the unit circle $|z|<1$ into D converges uniformly on the circle $|z|=1$; hence that there exists a continuous strictly increasing function $t(\theta)$ $(t(0)=0,\ t(2\pi)=2\pi)$ such that the Fourier series of $F(\theta)=f(t(\theta))$ and of $G(\theta)=g(t(\theta))$ converge uniformly for $0\leqq\theta\leqq 2\pi$. Using this theorem, J. Pál [2] has proved that given any continuous function $\phi(t)$ of period 2π there exists a function $t(\theta)$ of the above described type such that the Fourier series of $\phi(t(\theta))$ converges everywhere, and uniformly in the interval $\delta\leqq\theta\leqq 2\pi-\delta$, for any positive δ. H. Bohr [3] has removed the restriction on the uniform convergence in Pál's theorem by proving that the function $t(\theta)$ can be chosen such that the Fourier series of $\phi(t(\theta))$ converges uniformly for $0\leqq\theta\leqq 2\pi$. Bohr's argument involves some delicate considerations. The purpose of this paper is to give a short and simple proof of Bohr's result.

Let $\phi(t)$ be continuous, and of period 2π. Without loss of generality we can, by adding to ϕ a suitable constant, assume that $\int_0^{2\pi}\phi(t)dt=0$. Then there are values of t for which $\phi(t)$ vanishes and we can assume that $t\equiv 0 \pmod{2\pi}$ is one of these values. Thus $\phi(0)=\phi(2\pi)=0$. The mean value of the function being zero, there exists at least another point a $(0<a<2\pi)$ such that $\phi(a)=0$.

Suppose first that, in the open interval $(0, 2\pi)$, a is the only point at which $\phi(t)$ vanishes. Then $\phi(t)$ is strictly positive in one of the open intervals $(0, a)$, $(a, 2\pi)$, and strictly negative in the other one. Let $\alpha(t)$ be any function, continuous, of period 2π, such that $\alpha(0)=\alpha(2\pi)=0$ and such that $\alpha(t)$ is strictly increasing in $(0, a)$ and strictly decreasing in $(a, 2\pi)$. Then the equations $x=\alpha(t)$, $y=\phi(t)$ represent a simple closed Jordan curve and we have only to apply the theorem of Fejér quoted above to get our result for the function $\phi(t)$.

Suppose now that a is not the only point in the open interval $(0, 2\pi)$ at which $\phi(t)$ vanishes. Let M_1 be the maximum of $|\phi(t)|$ for $0\leqq t\leqq a$ and let t_1 be a point $(0<t_1<a)$ such that $|\phi(t_1)|=M_1$. In the same way let M_2 be the maximum of $|\phi(t)|$ in $a\leqq t\leqq 2\pi$ and let t_2 be

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Presented to the Society, February 26, 1944; received by the editors December 30, 1943.

[1] Numbers in brackets refer to the references cited at the end of the paper.

a point ($a < t_2 < 2\pi$) such that $|\phi(t_2)| = M_2$. Consider the function $\omega(t)$, continuous, of period 2π, and defined for every t ($0 \leqq t \leqq 2\pi$) as follows:

$$\omega(t) = \max_{0 \leqq t' \leqq t} |\phi(t')| + \sin(\pi t/a) \qquad \text{for } 0 \leqq t \leqq t_1,$$

$$\omega(t) = \max_{t \leqq t' \leqq a} |\phi(t')| + \sin(\pi t/a) \qquad \text{for } t_1 \leqq t \leqq a,$$

and in the same way

$$\omega(t) = -\max_{a \leqq t' \leqq t} |\phi(t')| - \sin[\pi(t-a)/(2\pi - a)] \qquad \text{for } a \leqq t \leqq t_2,$$

$$\omega(t) = -\max_{t \leqq t' \leqq 2\pi} |\phi(t')| - \sin[\pi(t-a)/(2\pi - a)] \qquad \text{for } t_2 \leqq t \leqq 2\pi,$$

where the maxima are taken with respect to t'.

It is immediately seen that $\omega(t)$ is of bounded variation, that $\phi_1(t) = \phi(t) + \omega(t)$ vanishes only at $t=0$, $t=a$, $t=2\pi$, for $0 \leqq t \leqq 2\pi$, and that $\phi_1(t)$ is strictly positive in the open interval $(0, a)$ and strictly negative in the open interval $(a, 2\pi)$. Hence applying our first result we can find a function $t(\theta)$ of the above described type such that the Fourier series of $\phi_1(t(\theta))$ converges uniformly for $0 \leqq \theta \leqq 2\pi$. But since $\omega(t)$ is continuous and of bounded variation the same result holds for $\phi(t)$, which proves the theorem.

References

1. L. Fejér, *La convergence sur son cercle de convergence d'une série de puissances effectuant une representation conforme du cercle sur le plan simple*, C. R. Acad. Sci. Paris vol. 156 (1913) pp. 46–49.

2. J. Pál, *Sur des transformations de fonctions qui font converger leurs séries de Fourier*, C. R. Acad. Sci. Paris vol. 158 (1914) pp. 101–103.

3. H. Bohr, *Über ein Satz von J. Pál*, Acta Univ. Szeged. vol. 7 (1935) pp. 129–135.

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