SUMMABILITY OF SUBSEQUENCES

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1. **Introduction.** Let a_{nk} $(n, k=1, 2, \cdots)$ be a matrix of real or complex constants for which

$$(1.1) \qquad \lim_{n\to\infty} a_{nk} = 0, \qquad k = 1, 2, 3, \cdots,$$

$$(1.2) \qquad \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1; \qquad \sum_{k=1}^{\infty} | a_{nk} | < M, \qquad n = 1, 2, 3, \cdots,$$

M being a constant. This matrix defines a regular method of summability by means of which a sequence x_n of real or complex numbers is summable to X if $X_n = \sum_{k=1}^{\infty} a_{nk}x_k$, $n=1, 2, 3, \cdots$, exists and $\lim X_n = X$. It has recently been shown by R. C. Buck[1] that if the sequence x_n is real, bounded, and divergent, then the sequence has a subsequence not summable A. This note proves the following more general theorem.

THEOREM. *Let A be regular and let x_n be a bounded complex sequence. Then there exists a subsequence y_n of x_n such that the set L_Y of limit points of the transform Y_n of y_n includes the set L_x of limit points of the sequence x_n.*

If x_n is a bounded divergent sequence, then L_x and hence also L_Y must contain at least two distinct points and accordingly the subsequence y_n is not summable A. Applying the theorem to the divergent sequence $0, 1, 0, 1, \cdots$, we obtain the result of Steinhaus[2] that there is a sequence of 0's and 1's not summable A.

2. **Proof of the theorem.** Let L_x be the set of limit points of the bounded complex sequence x_n. Since the complex plane is separable and L_x is a closed set, there is a countable (finite or infinite) subset E of L_x such that the closure $\bar{E}$ of E is the set L_x itself. Let $u_1, u_2, u_3, \cdots$ be a sequence containing all of the points of E; in case E is a finite set, the points $u_1, u_2, u_3, \cdots$ are not distinct. Let the elements of the sequence

$$(2.1) \qquad u_1;\ u_1, u_2;\ u_1, u_2, u_3;\ \cdots;\ u_1, u_2, \cdots, u_n;\ \cdots$$

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[1] R. C. Buck, *A note on subsequences*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 898–899.

[2] H. Steinhaus, *Some remarks on the generalization of limit* (in Polish), Prace Matematyczno-fizyczne vol. 22 (1911) pp. 121–134.

be denoted by $v_1, v_2, v_3, \cdots$. The sequence v_n is a sequence of points in L_x, and the set of limit points of the sequence is the set L_x. For each $p=1, 2, 3, \cdots$, let

$$x(n_{p1}),\ x(n_{p2}),\ x(n_{p3}),\ \cdots \tag{2.2}$$

be a subsequence of x_n having the limit v_p.

To simplify typography, we write $a(n, k)$ for a_{nk}. Since A is regular, (1.1) and (1.2) hold. Hence sequences $n_1<n_2<n_3<\cdots$ and $k_1<k_2<k_3<\cdots$ of indices exist such that for each $p=1, 2, 3, \cdots$

$$\sum_{k=1}^{k_p} | a(n_p, k) | < \frac{1}{p}, \qquad \sum_{k=k_{p+1}}^{\infty} | a(n_p, k) | < \frac{1}{p}. \tag{2.3}$$

It follows that

$$\begin{aligned} \sum_{k=k_p+1}^{k_{p+1}} a(n_p, k) &= \sum_{k=1}^{\infty} a(n_p, k) - \sum_{k=1}^{k_p} a(n_p, k) \\ &\quad - \sum_{k=k_{p+1}+1}^{\infty} a(n_p, k) = 1 + \epsilon_p \end{aligned} \tag{2.4}$$

where, here and hereafter, ϵ_p denotes generically a sequence for which $\epsilon_p \to 0$ as $p \to \infty$.

The subsequence $y(n)$ of the given sequence x_n is now selected as follows. Assuming that, for a fixed index p, $y(k)$ has been selected for each $k \leqq k_p$, let $y(k_p+1), \cdots, y(k_{p+1})$ be selected from the sequence (2.2) in such a way that $y(j)$ is a predecessor of $y(k)$ in the sequence x_n when $j<k$ and

$$| y(k) - v_p | < 1/p, \qquad k_p < k \leqq k_{p+1}. \tag{2.5}$$

Since x_n is bounded, say $|x_n| \leqq B$, the subsequence $y(n)$ thus defined by induction is bounded and accordingly possesses a transform

$$Y_n = \sum_{k=1}^{\infty} a_{n,k} y_k. \tag{2.6}$$

For each $p=1, 2, 3, \cdots$

$$\begin{aligned} Y(n_p) &= \sum_{k=1}^{k_p} a(n_p, k) y_k + \sum_{k=k_{p+1}+1}^{\infty} a(n_p, k) y_k + \sum_{k=k_p+1}^{k_{p+1}} a(n_p, k) y_k \\ &= \epsilon_p + \sum_{k=k_p+1}^{k_{p+1}} a(n_p, k) y_k, \end{aligned} \tag{2.7}$$

since each of the first two terms of the second member of (2.7) is dominated by B/p. Moreover

$$\begin{aligned}(2.8)\qquad \sum_{k=k_p+1}^{k_{p+1}} a(n_p, k)y_k &= v_p \sum_{k=k_p+1}^{k_{p+1}} a(n_p, k) + \sum_{k=k_p+1}^{k_{p+1}} a(n_p, k)(y_k - v_p) \\ &= v_p(1+\epsilon_p) + \epsilon_p = v_p + \epsilon_p.\end{aligned}$$

Therefore the sequences $Y(n_p)$ and v_p have the same limit points and accordingly the set of limit points of the sequence $Y(n_p)$ is identical with the set L_x of limit points of the sequence x_n. The set L_Y of limit points of the complete sequence Y_n therefore includes the set L_x and the theorem is proved.

3. **Conclusion.** It is apparent from the proof of the theorem that if x_n is a bounded divergent sequence, then it is possible to construct many subsequences y_n not summable A. However, the class of such subsequences y_n thus constructed seems to be a "small" subclass of the class of all subsequences of x_n. This observation is in agreement with the fact, recently proved by Buck and Pollard,[3] that if A is either convergence or the Cesàro method of order 1 and s_n is a real bounded sequence summable A, then there is a sense in which "almost all" of the subsequences of x_n are summable A.

The theorem states that the set L_Y of limit points of the transform Y_n of the subsequence y_n of x_n includes the set L_x of limit points of x_n. In some cases, L_Y is identical with L_x. This is so when A is convergence. In some cases, L_Y is more extensive than L_x. This is so when A is a Cesàro method of positive order and the set L_x is not connected since, as was shown by Barone,[4] the set of limit points of the transform of each bounded sequence must be connected. The same is true for methods of Hölder, Riesz, de la Vallee Poussin, and Euler.

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[3] R. C. Buck and Harry Pollard, *Convergence and summability properties of subsequences*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 924–931.

[4] H. G. Barone, *Limit points of subsequences and their transforms by methods of summability*, Duke Math. J. vol. 5 (1939) pp. 740–752.