BOOK REVIEW

Meromorphic functions and analytic curves. By Hermann Weyl and Joachim Weyl. (Annals of Mathematics Studies, no. 12.) Princeton University Press; London, Humphrey Milford and Oxford University Press, 1943. 9+269 pp. $3.50.

The theory of meromorphic functions began seriously with the now classical investigations of Poincaré, Hadamard, and Picard. After being continued by various other mathematicians it culminated in a paper by R. Nevanlinna in 1925, which was (according to the present author[1]) "one of the few great mathematical events in our century."

Nevanlinna associates with every meromorphic function $f(z)$ in the complex z-plane two non-negative real functions $N(r, a)$ and $m(r, a)$; the former counts how often f takes the value a in $|z| < r$, the latter measures the average closeness of f to a in $|z| < r$. Two of the principal results of Nevanlinna are these: (1) $N(r, a)+m(r, a)=T(r)+O(1)$, where $T(r)$ does not depend on a. (2) If any different complex numbers $a_1, \cdots, a_n$ and $\epsilon>0$ are given, then for all $r>0$ except for a set of finite measure, $\sum_i m(r, a_i)<(2+\epsilon)T(r)$. This implies in particular $N(r, a)\equiv 0$ for at most two a, that is, Picard's Theorem.

H. and J. Weyl and Ahlfors generalized these results in two respects. The underlying space is not any longer the z-plane, but any Riemann surface $\mathfrak{R}$, and the function f is replaced by $n+1$ functions on $\mathfrak{R}$ whose ratios define an analytic curve in the n-dimensional complex projective space P^n. Special cases are the algebraic curves (if $\mathfrak{R}$ is compact) and the meromorphic curves (if $\mathfrak{R}$ is the z-plane).

The first chapter provides the foundations for this theory and starts by introducing the Plücker coordinates of a $(p-1)$-dimensional linear subspace (p-spread) in P^n. The p-spreads appear as special vectors in the $C_{n,p}$-dimensional vector space of all p-ads. This space is metrized with the help of an Hermitian form in P^n. Next analytic curves are defined. An analytic curve C_1 determines for every p, $1<p\leqq n$, the locus C_p of its osculating p-spreads in the space of all p-ads. The methods of Hensel-Landsberg allow to treat the stationary indices of these curves and their relations to each other. This investigation leads for algebraic curves to the general Plücker relations between the orders and the stationary indices of the C_p.

The analogue to Nevanlinna's result (1) is the subject of chapter II. The function $N(r, a)$ is replaced by a function $N_p(R, A)$, $1\leqq p\leqq n$,

[1] The preface states that the book, while based on the joint work of the two authors, was actually written by H. Weyl.

which counts the number of intersections of C_p with the hyperplane A (in the space of the p-ads) for all points of C_p corresponding to values $|z| < R$. The function $m(r, a)$ becomes a function $m_p(R, A)$ which measures the average (unitary) distance of the same points from A. Then, independently of the unitary metric in P^n,

$$N_p(R, A) + m_p(R, A) = T_p(R) + O(1) \quad \text{(first main theorem).}$$

Meromorphic functions and exponential curves are treated as examples. The theorems of Poincaré and Hadamard on meromorphic functions of finite order conclude the chapter.

The following chapter (III) treats the relations between the order functions $T_p(R)$, $1 \leqq p \leqq n$, and the stationary indices of a given curve C_1, and yields thus an extension of the Plücker formulas. The result (second main theorem) is indeed surprisingly similar to these formulas. In addition we find several inequalities relating these numbers. One of the main tools is expressing the $T_p(R)$ as averages, for instance $T_1(R)$ turns out to be the average of $N_1(R, A)$ over all hyperplanes A_i of P^n.

In chapter IV the first and second main theorems are generalized to analytic curves. A system of bounded open sets G containing a fixed compact nucleus K_0 and exhausting the underlying Riemann surface R replaces the circles $|z| < R$. (In the case of meromorphic curves a fixed circle $|z| \leqq r_0$ plays the role of K_0; its influence is concealed in the $O(1)$ used by the reviewer.) The functions $N_p(G, A)$ are formed with the help of a harmonic function in $G - K_0$ which has the values 1 and 0 on the boundaries of K_0 and G respectively. The first main theorem holds except that $O(1)$ must be replaced by an expression $m^0(G, A)$ which depends on G and K_0 but whose order of magnitude can be estimated. A similar function $\eta(G, K_0)$ (and the genus of R) enters the second main theorem.

The last chapter deals with Picard's Theorem, more generally with Nevanlinna's defect relations. These results can be extended to analytic curves for which the above mentioned term $\eta(G, K_0)$ is small with respect to $T_1(G)$ (Hypothesis H). The implications of this assumption are rather difficult to understand. Whereas Picard's theorem for meromorphic functions on Riemann surfaces is readily obtained, the general defect relations are involved and require a great deal of technique. But they lead finally to the following beautiful result: If H holds then "for almost all" $G \supset K_0$ (compare the formulation of (2) above)

$$\sum_i m_p(G, A_i^p) < [C_{n,p} + \epsilon] T_p(G)$$

for any finite number of contravariant p-ads A_i^p in general position. This implies in particular that an analytic curve in P^n intersects at least one of any $n+1$ hyperplanes in general position.

The book begins with an allegoric preface and a historical introduction. The style is spirited and permeated by that elegant technique which the author displays in all his books. The reader sees always a wide mathematical landscape, and never just a one-dimensional approach to a given problem. Only in one instance, namely the construction of the harmonic function in chapter IV, §4, does it seem disputable whether the author chose the most efficient method. The reviewer has seen lecture notes on a course given by Ahlfors in Harvard which convinced him that subharmonic functions furnish a smoother approach, because they allow operating in the large.

It seems difficult to enumerate the exact prerequisites for the book. The reader must be familiar with various mathematical tools; this applies in particular to Riemann surfaces, since the indications in chapter I, §9, are not sufficient as a first introduction. The subject matter is involved and interrelated with several fields. The book is therefore not easy to read. But as always, the author rewards the reader by a wealth of information and mathematical technique which is useful far beyond the immediate scope of the book.

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