SUBDIRECT UNIONS IN UNIVERSAL ALGEBRA

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1. Preliminary definitions. By an *algebra*, we shall mean below any collection A of elements, combined by any set of single-valued operations f_α,

$$y = f_\alpha(x_1, \cdots, x_{n(\alpha)}). \tag{1}$$

The number of distinct operations (that is, the range of the variable α) may be infinite, but for our main result (Theorem 2), we shall require every $n(\alpha)$ to be finite—that is, it will concern *algebras with finitary operations*.

The concepts of *subalgebra*, *congruence relation* on an algebra, *homomorphism* of one algebra A onto (or into) another algebra with the same operations, and of the *direct union* $A_1\times \cdots \times A_r$ of any finite or infinite class of algebras with the same operations have been developed elsewhere.[1] More or less trivial arguments establish a many-one correspondence between the congruence relations θ_i on an algebra A and the homomorphic images $H_i=\theta_i(A)$ of the algebra (isomorphic images being identified); moreover the congruence relations on A form a lattice (the *structure lattice* of A). In this lattice, the equality relation will be denoted 0; all other congruence relations will be called *proper*.

More or less trivial arguments also show (cf. *Lattice theory*, Theorem 3.20) that the isomorphic representations of any algebra A as a *subdirect union*, or subalgebra $S\leqq H_1\times \cdots \times H_r$ of a direct union of algebras H_i, correspond essentially one-one to the sets of congruence relations θ_i on A such that $\wedge\theta_i=0$. In fact, given such a set of θ_i, the correspondence

$$\theta: \quad a \rightarrow [\theta_1(a), \cdots, \theta_r(a)] = [h_1, \cdots, h_r] \tag{2}$$

exhibits the desired isomorphism of A with a subalgebra of $H_1\times \cdots \times H_r$, where $H_i=\theta_i(A)$. Incidentally, the number of S_i can

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[1] *On the structure of abstract algebras*, Proc. Cambridge Philos. Soc. vol. 31 (1935) pp. 433–454, and in the foreword to the author's *Lattice theory*. The idea of an abstract congruence relation is also developed in chap. VI, §14, of S. MacLane's and the author's *Survey of modern algebra*. Interesting remarks in this connection may be found in J. C. C. McKinsey's and A. Tarski's *The algebra of topology*, Ann. of Math. vol. 45 (1944) esp. pp. 190–191.

be infinite. What is more important, the operations of A need not even be finitary.[2]

Equally trivial arguments extend a well known theorem of Emmy Noether on commutative rings[3] to abstract algebras in general. In order to state this extension, we first define an algebra A to be *subdirectly irreducible* if in any finite or infinite representation (2), some θ_i is itself an isomorphism. This means that the meet θ^* of all proper congruence relations on A is itself a proper congruence relation. In lattice-theoretic language, it means that the structure lattice $L(A)$ of A contains a *point* $\theta^*>0$ such that $\theta>0$ implies $\theta\geqq\theta^*$. Hence if A is subdirectly irreducible, $\theta\cap\theta'=0$ in $L(A)$ implies $\theta=0$ or $\theta'=0$; such an A we shall call *weakly irreducible.* If $L(A)$ satisfies the *descending* chain condition, and A is weakly irreducible, then it is evidently also subdirectly irreducible in the strong sense.

If $L(A)$ satisfies the *ascending* chain condition, then it is evident by induction[3] that there exists a representation of 0 as the meet $0=\theta_1\cap\cdots\cap\theta_r$ of a finite number of irreducible elements. This yields the following easy generalization of Emmy Noether's theorem on commutative rings.

THEOREM 1. *Any algebra A whose structure lattice satisfies the ascending chain condition is isomorphic with a subdirect union of a finite number of weakly irreducible algebras.*

For this result, we still do not need to assume that A has finitary operations.

2. **Main theorem.** Our principal result is the partial extension of Theorem 1 to algebras *without* chain condition. As will be seen in §3, this result will contain as special cases many known theorems and some new theorems.

THEOREM 2. *Any algebra A with finitary operations is isomorphic with a subdirect union of subdirectly irreducible algebras.*

PROOF. For any $a\neq b$ of A, consider the set $S(a, b)$ of all congruence relations θ on A, such that $a\not\equiv b \pmod{\theta}$. If T is any linearly ordered subset of $S(a, b)$, the *union* τ of the $\theta\in T$ is defined by the rule

$$(3)\qquad x\equiv y \pmod{\tau} \text{ means } x\equiv y \pmod{\theta} \text{ for some } \theta\in T.$$

It is evident that $a\not\equiv b \pmod{\tau}$, and that *if A has finitary operations,*

[2] This is observed in N. H. McCoy and Deane Montgomery, *A representation of generalized Boolean rings*, Duke Math. J. vol. 3 (1937) p. 46, line 12.

[3] Cf. van der Waerden, *Moderne Algebra*, first ed., vol. 2, p. 36. The unicity theorem on p. 40 does not apply to abstract algebras in general, however.

then τ is a congruence relation. Hence, in the structure lattice $L(A)$ of A, the union of any linearly ordered subset of $S(a, b)$ exists and is in $S(a, b)$. But this is the hypothesis of the "first form" of Zorn's Lemma.[4] The conclusion is that $S(a, b)$ contains a *maximal* element, $\theta_{a,b}$. We next consider $H_{a,b}$, the homomorphic image of A, mod $\theta_{a,b}$.

Every proper congruence relation $\theta > 0$ corresponds[5] to a $\theta' > \theta_{a,b}$; and since $\theta_{a,b}$ is maximal in $S(a, b)$, this implies $a \equiv b \pmod{\theta'}$. Hence the *meet* θ^* of the $\theta > 0$ in $H_{a,b}$, defined by

$$(4) \qquad x \equiv y \pmod{\theta^*} \text{ means } x \equiv y \pmod{\theta} \text{ for all } \theta > 0,$$

will satisfy $a \equiv b \pmod{\theta^*}$, and hence $\theta^* > 0$. Hence (cf. §1) $H_{a,b}$ is subdirectly irreducible.

Finally, the meet of *all* $\theta_{a,b}$ is 0, since we have identically $x \not\equiv y \pmod{\theta_{x,y}}$. Hence, by the theorem cited in §1, paragraph 3, A is isomorphic with a subdirect union of the (subdirectly irreducible) $H_{a,b}$, q.e.d.

3. **Applications.** Theorem 2 has importance mainly because subdirectly irreducible algebras may be specifically described in so many cases.

LEMMA 1. *A weakly irreducible distributive lattice or Boolean algebra must consist of* 0 *and* I *alone.*

PROOF FOR DISTRIBUTIVE LATTICES. For any a, the endomorphisms $\theta_a : x \rightarrow x \cup a$ and $\theta_a' : x \rightarrow x \cap a$ have the property[6] that $\theta_a \cap \theta_a' = 0$, and neither defines the equality relation unless $a = 0$ or $a = I$.

PROOF FOR BOOLEAN ALGEBRAS. Let $x \equiv y \pmod{\theta_a}$ mean $|x - y| \leqq a$ (symmetric difference notation); then $\theta_a \cap \theta_{a'} = 0$, and neither defines the equality relation unless $a = 0$ or $a' = 0$ $(a = I)$.

COROLLARY 1. *Any distributive lattice is isomorphic with a ring of sets.*[7]

COROLLARY 2. *Any Boolean algebra is isomorphic with a field of sets.*[7]

[4] Cf. J. W. Tukey, *Convergence and uniformity in topology*, Princeton, 1940, p. 7.

[5] We omit discussing the obvious isomorphism between $L(H_{a,b})$ and the sublattice of $\theta' > \theta_{a,b}$ in $L(A)$.

[6] PROOF. If $x \cup a = y \cup a$ and $x \cap a = y \cap a$, then $x = x \cap (x \cup a) = x \cap (y \cup a) = (x \cap y) \cup (x \cap a) = (y \cap x) \cup (y \cap a) = y \cap (x \cup a) = y \cap (y \cup a) = y$—by the distributive laws. θ_a and θ_a' are endomorphisms.

[7] Corollaries 1–2 are theorems of the author and Stone, respectively. Corollary 3 below is due to McCoy and Montgomery, op. cit. footnote 2. Corollary 4 is due to G. Kothe, *Abstrakte Theorie nichtkommutative Ringe*, Math. Ann. vol. 103 (1930) p. 552; N. H. McCoy *Subrings of infinite direct sums*, Duke Math. J. vol. 4 (1938) pp. 486–494.

LEMMA 2. *A subdirectly irreducible commutative ring R without nilpotent elements is a field.*[8]

PROOF. As in §1, R will have a unique minimal ideal (that is, congruence relation) J. But for any $a\neq 0$ in J, since $aa\neq 0$, $aJ>0$. Moreover since $(aJ)R=a(JR)\leqq aJ$, aJ is an ideal, $0<aJ\leqq J$. Consequently $aJ=J$—whence J is a *field* (Huntington's postulates) with unit e. The set $0:e$ of all $x\in R$ such that $ex=0$ is an ideal, and $(0:e)\cap J=0$ by what we have just shown; hence $0:e=0$. But for any $x\in R$, $e(x-ex)=0$; hence $(x-ex)\in 0:e=0$, and $x=(x-ex)+ex$, $0=ex\in J$. We conclude that $R=J$ is a field, q.e.d.

One might infer that by Theorem 2 any commutative ring without nilpotent elements was a subdirect product of fields, but this reasoning would be invalid. It is not necessarily true that a homomorphic image of rings without nilpotent elements is itself without nilpotent elements.

On the other hand, any homomorphic image of a *p-ring* (or commutative ring in which $a^p=a$ for some prime p) is itself a p-ring, and evidently without nilpotent elements, since $a^{p^n}=a$ for all n. Furthermore, a field in which $a^p=a$ can contain only p elements, and must be $GF(p)$ (or the "field" 0).

COROLLARY 3. *Any p-ring is a subdirect union of $GF(p)$, or consists of* 0 *alone.*[7]

Again, any homomorphic image of a *regular* ring in the sense of von Neumann (or ring in which any a has a "relative inverse" u such that $aua=a$) is itself regular, and evidently without nilpotent elements if commutative (since $a^nu^{n-1}=auau\cdots ua=a\neq 0$).

COROLLARY 4. *Any commutative "regular" ring is a subdirect union of fields.*[7]

If one were interested in obtaining corollaries of Theorem 1, one might show that even a *weakly* irreducible p-ring or regular ring was a field. Again, one might show (van der Waerden, op. cit. p. 32) that, in a weakly irreducible commutative ring satisfying the chain condition, every divisor of zero is nilpotent; this would yield E. Noether's theorem that every commutative ring satisfying the chain condition was a subdirect union of a finite number of primary rings.

Similarly, one can show easily that the only weakly irreducible vector space over a field F is the one-dimensional vector space $V(F;1)$ (or 0). It follows that any vector space is a subdirect union of one-

[8] This lemma was suggested to the author in conversation by N. H. McCoy.

dimensional vector spaces. Actually (due to the existence of bases) a stronger result is well known.

LEMMA 3. *The only weakly irreducible commutative groups G are the "generalized cyclic" groups: the additive subgroups of the rationals, and those of the rationals* mod *one.*

We omit the proof, which follows easily from the fact that a commutative group with two generators is cyclic unless it contains two disjoint subgroups (the latter hypothesis would make G weakly reducible).

COROLLARY 5. *Any commutative group is a subdirect union of generalized cyclic groups.*

The center of any weakly irreducible hypercentral (alias nilpotent) group H is generalized cyclic (the proof is trivial, granted Lemma 3); the converse also holds if H is finite.[9] Hence we have the following corollary.

COROLLARY 6. *Any hypercentral group is a subdirect union of groups with generalized cyclic centers.*

Further, any weakly irreducible commutative l-group (lattice-ordered group) is known[10] to be *simply* ordered. This yields the following corollary.

COROLLARY 7. *Any commutative l-group is a subdirect union of simply ordered* 1*-groups.*

One can easily show (although we omit the proof) that any closed element in a closure algebra (in the sense of McKinsey and Tarski[11]) determines a congruence relation, essentially through relativization with respect to the complementary open set. Then from the definition of well-connectedness one obtains the following corollary.

COROLLARY 8. *Any "closure algebra" is a subdirect union of "well-connected" closure algebras.*

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[9] Theorems of Burnside (cf. H. Zassenhaus, *Gruppentheorie*, Teubner, 1937, p. 107, Satz. 11) and P. Hall, *A contribution to the theory of groups of prime-power orders*, Proc. London Math. Soc. vol. 36 (1933) p. 51, Theorem 2.49.

[10] Cf. the author's *Lattice-ordered groups*, Ann. of Math. vol. 43 (1942) p. 319.

[11] *The algebra of topology*, Ann. of Math. vol. 45 (1944) pp. 141–191. The definition of well-connectedness is on p. 147; the concept of relativization is developed on p. 151.