BOOK REVIEW

Integration. By E. J. McShane. Princeton University Press, 1944. 8+392 pp. \$6.00.

Modern mathematical analysis deals with many different methods of assigning values to integrals. If A and B are two such methods, perhaps those of Riemann and Lebesgue, we may use the symbols

$$(1) \qquad (A)\int_E f(x)dx, \qquad (B)\int_E f(x)dx$$

to denote the corresponding values of the integrals when E and $f(x)$ are such that the integrals exist, that is, when E and $f(x)$ are such that the methods assign values to the integrals. The following terminology, while not so firmly established in the theory of integration as in the theory of summability of infinite series, serves as a basis for precise statements in the theory of integration. Two methods A and B for evaluation of integrals are *consistent* if the two integrals in (1) are equal whenever both exist. The two methods A and B are *equivalent* if the two integrals in (1) both exist and are equal whenever E and $f(x)$ are such that at least one of the two exists. A method A *includes* a method B if existence of the second integral in (1) implies existence of the first and equality of the two. Thus A and B are equivalent if and only if each includes the other. The esteem attached to the notion of regularity in the theory of summability suggests that a method for evaluation of integrals should be called *regular* if it includes the Riemann method.

It could be argued that, in a textbook in applied mathematics in which there is a tacit understanding that only integrals consistent with each other are employed, there is no point in bothering with names of integrals. Whether the integrals be Riemann, Darboux, Lebesgue, Cauchy-Riemann, Cauchy-Lebesgue, Riemann-Stieltjes, and so on, does not matter. In such a book, the statement that a function is integrable means simply that it is integrable by one of the methods in the (undefined) family used. When an integrable function is given, one evaluates or estimates the integral, and proceeds to use the result. In such a book, the statement that a given function is not integrable could not be proved; there is always the possibility that there is some method in the undefined family by means of which the integral can be evaluated. There is no reason why one must stop when he has reached a general Banach integral by means of which every bounded function is integrable over every bounded interval over which it is defined.

A textbook on the theory of integration should, as the one under review does, make careful presentations of its definitions of integral. It should, as the one under review does, give proper attention to questions of consistency. It should, as the one under review does to some extent, make distinctions between methods which are not equivalent. It should, as the one under review fails to do in important cases pointed out below, make distinctions between equivalent methods which are conceptually different. That the author recognizes the desirability of making these distinctions is clearly indicated on p. 54 where he wrote "The reason for marking the integral sign with a prime in the definition is to avoid its confusion with other integrals later to be defined; even when applied to step-functions these later (and more useful) definitions will be conceptually different . . . and may even be numerically different."

This book deals mainly with Lebesgue and Lebesgue-Stieltjes integrals and with integrals equivalent to them. The author says in his preface that it is designed for students of little maturity. While less detailed, its style is similar to that of Hobson's *Theory of functions of a real variable.*

Chap. 1, pp. 1–51, is an introductory chapter giving a precise development of the fundamentals of the theory of sets in R_q (Euclidean space of q dimensions) and of real functions defined over such sets.

In chap. 2, pp. 52–100, Lebesgue integrals are defined, in terms of bounds of other integrals, without mention of measurable sets and measurable functions. The volume ΔI of an interval (or cell) I in R_q is defined to be the product of its dimensions. It is in accordance with the tenor of the book that detailed proofs of such theorems as the following are given. If $I_1, I_2, \cdots, I_n$ are closed non-overlapping intervals whose sum (union) is an interval I, then $\sum \Delta I_k = \Delta I$. Let I be an interval which is the union of disjoint subintervals $I_1, I_2, \cdots, I_n$, let $c_1, \cdots, c_n$ be constants, and let $s(x)$ be the step function for which $s(x) = c_k$ when x is in I_k. The *prime* integral of $s(x)$ over I is defined by

$$\int_I' s(x)dx = \sum_{k=1}^{n} c_k | I_k |.$$

A few properties of the prime integral are developed. The upper and lower Darboux (the author says Riemann) integrals over I of a function $f(x)$ bounded over I are defined to be, respectively, the greatest lower bound of $\int_I' s(x)dx$ for all step functions $s(x) \geqq f(x)$ over I, and the least upper bound of the same integral for all step functions $s(x) \leqq f(x)$ over I. This is a neat way of phrasing the classic definitions of Darboux. When these upper and lower integrals are equal, their common value (the Darboux integral) is called the Riemann integral. There is no mention of the Riemann sums used by Riemann to define his integral. Some properties of the Riemann integral are developed. Special integrals of semicontinuous functions are defined in terms of bounds of integrals of continuous functions in specified classes. Special upper and lower integrals of functions $f(x)$ defined over a finite closed interval I are defined in terms of bounds of integrals of semicontinuous functions in specified classes. If the latter upper and lower integrals are equal, the author defines their common value to be the Lebesgue integral of $f(x)$ over I. Just as the Darboux concept of integral is different from that of Riemann, so also this concept of integral is different from that of Lebesgue. It is not proved that this method of assigning values to integrals is equivalent to that of Lebesgue; unfortunately, the book does not mention Lebesgue's fundamental concept of the value of an integral. The nearest approach to this matter comes on p. 125 where it is shown that if $f(x)$ is bounded and measurable over a set E of finite measure, then $f(x)$ is Lebesgue integrable over E. At this point, it would be easy to give Lebesgue's own definition of integral and use it to prove the required equivalence. On the basis of the author's definition, the fundamental properties of the Lebesgue integral are derived.

In chap. 3, pp. 101–135, the theory of measurable sets is obtained by applying the theory of integration to characteristic functions. The theory of measurable functions is then developed. The classes L_p are defined, and the Hölder and Minkowski inequalities are proved. Chap. 4, pp. 136–187, gives the Fubini and other theorems on iterated integrals. Functions of sets are defined and discussed. Fatou's lemma is proved. Chap. 5, pp. 188–217, deals with differentiation of integrals and change of variable of integration. Chap. 6, pp. 218–241, deals with approximations to functions. In Chap. 7, pp. 242–311, Lebesgue-Stieltjes integrals are developed for functions defined over sets in R_q. Chap. 8, pp. 312–335, develops the Perron integral. Chap. 9, pp. 336–365, gives existence theorems for systems of differential equations. The final chapter, Chap. 10, pp. 366–382, deals with differentiation of multiple integrals. There is a good index.

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