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## A GENESIS FOR CESÀRO METHODS

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1. **Introduction.** The Cesàro methods  $C_r$ , introduced by Cesàro<sup>1</sup> because of their applicability to Cauchy products of series, constitute the most publicized class of methods of summability.

The regular Nörlund methods<sup>2</sup> of summability constitute one of the two most publicized general classes of consistent methods of summability. The regular Hurwitz-Silverman-Hausdorff methods constitute the other.

This note proves the following theorem.

**THEOREM.** *The Cesàro methods are the only methods of summability, regular or not, which are both Nörlund methods and Hurwitz-Silverman-Hausdorff methods.*

Thus if the Cesàro methods had not been previously introduced into mathematical literature, they could be defined and exploited as the unique class of methods of summability enjoying all of the properties of Nörlund methods and all of the properties of Hurwitz-Silverman-Hausdorff methods.

In §4, it is shown that the only methods which are both Riesz methods and Hurwitz-Silverman-Hausdorff methods are methods  $\Gamma_r$ , closely related to the methods  $C_r$ .

2. **Nörlund methods.** Each sequence  $p_0, p_1, \dots$  of real or complex constants for which  $P_n \equiv p_0 + p_1 + \dots + p_n \neq 0$  for each  $n$  defines a Nörlund method of summability by means of which a sequence  $s_0, s_1, \dots$  is summable to  $\sigma$  if  $\sigma_n \rightarrow \sigma$  where

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<sup>1</sup> E. Cesàro, *Sur la multiplication des series*, Bull. Sci. Math. (2) vol. 14 (1890) pp. 114–120.

<sup>2</sup> N. E. Nörlund, *Sur une application des fonctions permutables*, Lunds Universitets Arsskrift (2) vol. 16 (1919).

$$(1) \quad \sigma_n = (p_n s_0 + p_{n-1} s_1 + \cdots + p_1 s_{n-1} + p_0 s_n) / P_n.$$

The class of Nörlund transformations (1) is identical with the class of triangular matrix transformations

$$(2) \quad \sigma_n = \sum_{k=0}^n a_{nk} s_k$$

for which

$$(3) \quad a_{nn} \neq 0, \quad n = 0, 1, 2, \dots,$$

$$(4) \quad \sum_{k=0}^n a_{nk} = 1, \quad n = 0, 1, 2, \dots,$$

and, for each  $q=0, 1, 2, \dots$ , there is a constant  $b_q$  such that

$$(5) \quad a_{n,n-q} = b_q a_{nn}, \quad n \geq q.$$

**3. Hurwitz-Silverman-Hausdorff methods.** These methods (hereafter the HSH methods) constitute the class of triangular matrix transformations which commute with the arithmetic mean transformation

$$(6) \quad \sigma_n = (s_0 + s_1 + \cdots + s_n) / (n + 1)$$

and hence also with each other.

As Hurwitz and Silverman<sup>3</sup> and Hausdorff<sup>4</sup> have shown, with each method HSH there is associated a generating sequence  $\lambda_0, \lambda_1, \dots$  such that the transformation takes the form

$$(7) \quad \sigma_n = \sum_{j=0}^n (-1)^j C_{n,j} \lambda_j \sum_{k=0}^j (-1)^k C_{j,k} s_k$$

or

$$(8) \quad \sigma_n = \sum_{k=0}^n \left[ C_{n,k} \sum_{j=0}^{n-k} (-1)^j C_{n-k,j} \lambda_{j+k} \right] s_k.$$

Assume that (8) is a Nörlund transformation, and let the quantity in brackets in (8) be denoted by  $a_{nk}$ . Then,  $a_{nn} = \lambda_n$  so, by (3) and (4),  $\lambda_n \neq 0$  for each  $n$  and  $\lambda_0 = 1$ . Moreover

$$(9) \quad a_{n,n-1} = n(\lambda_{n-1} - \lambda_n)$$

<sup>3</sup> W. A. Hurwitz and L. L. Silverman, *On the consistency and equivalence of certain definitions of summability*, Trans. Amer. Math. Soc. vol. 18 (1917) pp. 1-20.

<sup>4</sup> F. Hausdorff, *Summationsmethoden und Momentfolgen*. I and II, Math. Zeit. vol. 9 (1921) pp. 74-109 and 280-299.

and hence (5) with  $q=1$  guarantees existence of a constant (which we call  $r$  instead of  $q_1$ ) such that

$$(10) \quad n(\lambda_{n-1} - \lambda_n) = r\lambda_n, \quad n \geq 1.$$

Here  $r$  cannot be a negative integer; otherwise one could set  $n = -r$  in (10) and contradict  $\lambda_{-r-1} \neq 0$ . Therefore

$$(11) \quad \lambda_n = (n/(n+r))\lambda_{n-1}, \quad n = 1, 2, \dots$$

Since  $\lambda_0=1$ , (11) implies that

$$(12) \quad \lambda_1 = 1/(1+r), \quad \lambda_2 = (2/(2+r))\lambda_1 = 1 \cdot 2/(1+r)(2+r)$$

and in general

$$(13) \quad \lambda_n = 1 \cdot 2 \cdot 3 \cdots n/(1+r)(2+r) \cdots (n+r) = n!r!/(n+r)!, \quad n = 0, 1, \dots,$$

where  $r!$  is the factorial function,  $\Gamma(r+1)$ , defined for all complex  $r$  except  $-1, -2, -3, \dots$ .

By use of the familiar identity

$$(14) \quad \sum_{k=0}^n C_{n-k+r-1, r-1} = C_{n+r, r}, \quad r \neq -1, -2, \dots,$$

we obtain the familiar fact that for each  $r \neq -1, -2, \dots$  the Cesàro transformation  $C_r$ ,

$$(15) \quad \sigma_n = \sum_{k=0}^n [C_{n-k+r-1, r-1}/C_{n+r, r}]s_k,$$

is the one and only Nörlund transformation for which  $a_{nn}=\lambda_n = n!r!/(n+r)!$ . This completes the proof of the fact that the only HSH transformations which are Nörlund transformations are the Cesàro transformations.

It is only when  $r=0$  or  $r$  is a complex number with a positive real part that  $C_r$  is regular and can be written in the Hausdorff form

$$(16) \quad \sigma_n = \int_0^1 \sum_{k=0}^n C_{n, k} t^k (1-t)^{n-k} s_k d\chi(t)$$

where

$$(17) \quad \chi(t) = 1 - (1-t)^r.$$

For each complex  $r$  not a negative integer, the more general Hurwitz-Silverman transformation (8) becomes  $C_r$  when  $\lambda_n = n!r!/(n+r)!$ .

4. **Riesz methods.** Each sequence  $p_0, p_1, \dots$  for which  $P_n \equiv p_0 + p_1 + \dots + p_n \neq 0$  for each  $n$  determines a Riesz transformation

$$(18) \quad \sigma_n = (p_0 s_0 + p_1 s_1 + \dots + p_n s_n) / P_n, \quad n = 0, 1, 2, \dots$$

If one of the two transformations

$$(19) \quad \sigma'_n = \sum_{k=0}^n a_{nk} s_k, \quad \sigma''_n = \sum_{k=0}^n a_{n,n-k} s_k$$

is a Nörlund transformation, the other is a Riesz transformation. It can be shown, by a suitable modification of our treatment of HSH and Nörlund methods, that if  $A$  is both an HSH method and a Riesz method, then there is a constant  $r$  not a negative integer such that  $A$  has the form

$$(20) \quad \sigma_n = \sum_{k=0}^n [C_{k+r-1, r-1} / C_{n+r, r}] s_k.$$

Thus *the only methods which are simultaneously HSH methods and Riesz methods are the methods  $\Gamma_r$  defined by (20)*. For each  $r \neq -1, -2, \dots$ , the transformation  $\Gamma_r$  is obtained from  $C_r$  by reversing the order in which the elements  $a_{nk}^{(r)}$  are applied to  $s_0, s_1, \dots$ . The HSH sequence  $\lambda_n$  generating  $\Gamma_r$  is  $\lambda_0 = a_{00} = 1$  and

$$(21) \quad \lambda_n = a_{nn} = r / (n + r), \quad n > 0.$$

The following discussion applies only to *regular* transformations. The elements  $a_{nk}$  of the matrix of the Hausdorff transformation generated by  $\chi(t)$  are given by

$$(22) \quad a_{n,k} = \int_0^1 C_{n,k} t^k (1-t)^{n-k} d\chi(t).$$

If (22) holds, then

$$\begin{aligned} a_{n,n-k} &= \int_0^1 C_{n,k} t^{n-k} (1-t)^k d\chi(t) \\ &= - \int_0^1 C_{n,k} u^k (1-u)^{n-k} d_u \chi(1-u) \end{aligned}$$

and hence

$$(23) \quad a_{n,n-k} = \int_0^1 C_{n,k} t^k (1-t)^{n-k} d_t [1 - \chi(1-t)].$$

It follows that if  $\chi(t)$  generates one of the transformations in (19), then  $\chi_1(t) \equiv 1 - \chi(1-t)$  generates the other. Suppose now that  $\chi(t)$

generates a regular Riesz method. Then  $1 - \chi(1-t)$  generates a Nörlund method, which must be a Cesàro method  $C_r$ . Hence  $1 - \chi(1-t) = 1 - (1-t)^r$  and  $\chi(t) = t^r$ . Since  $t^r$  generates a regular HSH method only when  $\Re r > 0$ , and since  $t^r$  generates the regular Riesz method  $\Gamma_r$  when  $\Re r > 0$ , we have proved the following result. *The methods  $\Gamma_r$  for which  $\Re r > 0$  are the only regular methods of summability which are simultaneously Riesz methods and HSH methods.*

The identity

$$(24) \quad \frac{r!n!}{(n+r)!} = \frac{(r-1)!n!}{(n+r-1)!} \frac{r}{n+r}, \quad r \neq 0, -1, \dots,$$

implies the identities  $C_r = C_{r-1}\Gamma_r = \Gamma_r C_{r-1}$  and  $\Gamma_r = C_r C_{r-1}^{-1} = C_{r-1}^{-1} C_r$  involving the methods  $\Gamma_r$  and  $C_r$ ; this is a consequence of the fact (Hurwitz-Silverman, loc. cit. p. 7) that if  $\lambda_n'$  generates  $A'$  and  $\lambda_n''$  generates  $A''$ , then  $\lambda_n' \lambda_n''$  generates  $A'A''$ . From the fact that  $C_r$  and the Hölder method  $H_r$  are equivalent ( $C_r \sim H_r$ ) when  $\Re r > -1$  we obtain, when  $\Re r > 0$ , the familiar formulas

$$(25) \quad C_r \sim H_r = H_{r-1}H_1 \sim C_{r-1}H_1 = C_1 C_{r-1}$$

and  $C_r C_{r-1}^{-1} \sim C_1$ . This gives the fact, proved by Hausdorff, loc. cit., that  $\Gamma_r$  is equivalent to  $C_1$  when  $\Re r > 0$ .

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