A COMBINATORIAL FORMULA WITH SOME APPLICATIONS

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The aim of this note is to present a combinatorial formula and state its applications to partitions, number of solutions, and Dirichlet's integral.

Let $\theta_1(x), \cdots, \theta_n(x)$ be n arbitrary functions of x and let ${}_m\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}([\theta_1]\cdots[\theta_n])$ be defined by

$$(1)\qquad {}_m\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}([\theta_1]\cdots[\theta_n]) = \sum_{x_1+\cdots+x_n=m,\, a_1\leqq x_1\leqq b_1\cdots a_n\leqq b_n} \theta_1(x_1)\cdots\theta_n(x_n),$$

where $a_1, \cdots, a_n, b_1, \cdots, b_n, m$ are all integers and the right-hand side of (1) is summed over all different integral solutions of $x_1+\cdots+x_n=m$ with $a_1\leqq x_1\leqq b_1, \cdots, a_n\leqq x_n\leqq b_n$.

More generally we define

$$(2)\qquad \begin{aligned} &{}_m\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}([\theta_1]^{n_1}\cdots[\theta_k]^{n_k}) \\ &= \sum_{x_{11}+\cdots+x_{1n_1}+\cdots+x_{k1}+\cdots+x_{kn_k}=m,\, a_1\leqq x_{1i}\leqq b_1,\cdots,a_k\leqq x_{ki}\leqq b_k} \theta_1(x_{11}) \\ &\qquad \cdots\theta_1(x_{1n_1})\cdots\theta_k(x_{k1})\cdots\theta_k(x_{kn_k}). \end{aligned}$$

We make the following conventions:[1]

(A) ${}_m\mathfrak{S}_a^b([\theta]^n)=0$ for $m<na$ or $m>nb$.

(B) ${}_m\mathfrak{S}([\theta]^0)=0$ if $m\neq 0$, ${}_m\mathfrak{S}([\theta]^0)=1$ if $m=0$.

(C) If $a_1=\cdots=a_n=a$, $b_1=\cdots=b_n=b$, we write

$${}_m\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}([\theta_1]\cdots[\theta_n]) \quad \text{as} \quad {}_m\mathfrak{S}_a^b([\theta_1]\cdots[\theta_n]).$$

We now show that[2]

$$(3)\qquad \begin{aligned} &\sum_{\nu_1=0}^{n_1}\sum_{\nu_2=0}^{n_2}\cdots\sum_{\nu_k=0}^{n_k}(-1)^{\nu_1+\nu_2+\cdots+\nu_k}C_{n_1,\nu_1}C_{n_2,\nu_2}C_{n_k,\nu_k} \\ &\qquad \cdot{}_{m'}\mathfrak{S}_1^{\infty}([\phi_1]^{\nu_1}[\psi_1]^{n_1-\nu_1}\cdots[\phi_k]^{\nu_k}[\psi_k]^{n_k-\nu_k}) \\ &\qquad = {}_m\mathfrak{S}_{a_1\cdots a_k}^{b_1\cdots b_k}([\theta_1]^{n_1}\cdots[\theta_k]^{n_k}), \end{aligned}$$

where $\theta_1, \cdots, \theta_k$ are k arbitrary functions of x and $\phi_i(x)=\theta_i(x+b_i)$,

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[1] These conventions are used in proving (3) and other formulas.

[2] The formula (3) shows that all the strong restrictions for x can be removed.

$\psi_i(x)=\theta_i(x+a_i-1)$, $m'=m'(\nu_1\cdots\nu_k)=m-\sum b_i\nu_i-\sum(a_i-1)(n_i-\nu_i)$ $(i=1,\cdots,k)$.

PROOF. By definition we have

$$\begin{aligned}{}_{m'}\mathfrak{S}_1^\infty([\phi_1]^{\nu_1}[\psi_1]^{n_1-\nu_1}\cdots[\phi_k]^{\nu_k}[\psi_k]^{n_k-\nu_k}) &= \sum_{m_1+\cdots+m_{2k}=m} {}_{m_1}\mathfrak{S}_{b_1+1}^\infty([\theta_1]^{\nu_1})\,{}_{m_2}\mathfrak{S}_{a_1}^\infty([\theta_1]^{n_1-\nu_1}) \\ &\qquad\cdots{}_{m_{2k-1}}\mathfrak{S}_{b_k+1}^\infty([\theta_k]^{\nu_k})\,{}_{m_{2k}}\mathfrak{S}_{a_k}^\infty([\theta_k]^{n_k-\nu_k}).\end{aligned}$$

Let $T_i=\theta_i(x_{i1})\cdots\theta_i(x_{in_i})$ $(i=1,\cdots,k)$, and let $T=T_1\cdots T_k$ be a term contained in the right-hand side of (3), that is, $x_1\geqq a_1,\cdots,x_k\geqq a_k$. Without loss of generality we may assume $x_{11},\cdots,x_{1t_1}\geqq b_1+1;\cdots;x_{k1},\cdots,x_{kt_k}\geqq b_k+1$. Since the necessary and sufficient condition for

$$T_i\in{}_{m_{2i-1}}\mathfrak{S}_{b_i+1}^\infty([\theta_i]^{\nu_i})\,{}_{m_{2i}}\mathfrak{S}_{a_i}^\infty([\theta_i]^{n_i-\nu_i})\qquad(1\leqq i\leqq k)$$

is that there is a term $\theta_i(x_1)\cdots\theta_i(x_{\nu_i})$ of ${}_{m_{2i-1}}\mathfrak{S}_{b_i+1}^\infty$ contained in T_i as a part while the other part $T_i/\theta_i(x_1)\cdots\theta_i(x_{\nu_i})$ is contained in ${}_{m_{2i}}\mathfrak{S}_{a_i}^\infty$, the number of occurrences of T in the left-hand side of (3) is therefore given by

$$\left\{\sum_{\nu_1=0}^{t_1}(-1)^{\nu_1}C_{t_1,\nu_1}\right\}\left\{\sum_{\nu_2=0}^{t_2}(-1)^{\nu_2}C_{t_2,\nu_2}\right\}\cdots\left\{\sum_{\nu_k=0}^{t_k}(-1)^{\nu_k}C_{t_k,\nu_k}\right\} = \begin{cases}0 & \text{if } t_1,\cdots,t_k \text{ are not all zero,}\\ 1 & \text{if } t_1=\cdots=t_k=0.\end{cases}$$

We see that the term T generally vanishes except when $a_j\leqq x_{ji}\leqq b_j$ $(j=1,\cdots,k)$. Hence (3) is proved.

It is directly deduced from (3) by putting $n_1=\cdots=n_k=1$, $n_1+\cdots+n_k=n$ that

$$\text{(4)}\quad \begin{aligned}\sum_{k=0}^n(-1)^k\sum_{(\alpha_1\cdots\alpha_k)\in(1\cdots n)}{}_{m'}\mathfrak{S}_1^\infty([\phi_{\alpha_1}]\cdots[\phi_{\alpha_k}][\psi_{\alpha_{k+1}}]\cdots[\psi_{\alpha_n}]) \\ = {}_m\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}([\theta_1]\cdots[\theta_n]),\end{aligned}$$

where $(\alpha_1\cdots\alpha_k\cdots\alpha_n)=(1\cdots n)$, $m'=m'(\alpha_1\cdots\alpha_k)=m+n-k-(b_{\alpha_1}+\cdots+b_{\alpha_k}+a_{\alpha_{k+1}}+\cdots+a_{\alpha_n})$.

Let F be an arbitrary function of $\theta_1,\cdots,\theta_n$ and let ${}_m\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}\{F(\theta_1,\cdots,\theta_n)\}$ be defined by

$${}_m\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}\{F(\theta_1,\cdots,\theta_n)\} = \sum_{x_1+\cdots+x_n=m,\,a_1\leqq x_1\leqq b_1,\cdots,a_n\leqq x_n\leqq b_n}F(\theta_1(x_1),\cdots,\theta_n(x_n)).$$

Then, the formula (4) can be written more generally as

$$(5)\quad \sum_{k=0}^{n}(-1)^k \sum_{(\alpha_1\cdots\alpha_k)\in(1\cdots n)} {}_{m'}\mathfrak{S}_1^{\infty}\{F(\phi_{\alpha_1},\cdots,\phi_{\alpha_k},\psi_{\alpha_{k+1}},\cdots,\psi_{\alpha_n})\} = {}_{m}\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}\{F(\theta_1,\cdots,\theta_n)\}.$$

If, for every $(\alpha_1\cdots\alpha_k)$, the limit

$$\lim_{m\to\infty} {}_{m'}\mathfrak{S}_1^{\infty}\{F(\phi_{\alpha_1},\cdots,\phi_{\alpha_k},\psi_{\alpha_{k+1}},\cdots,\psi_{\alpha_n})\} \qquad (m'=m'(\alpha_1\cdots\alpha_k))$$

exists, then we have further

$$(6)\quad \sum_{k=0}^{n}(-1)^k \sum_{(\alpha_1\cdots\alpha_k)\in(1\cdots n)} \lim_{m\to\infty} {}_{m'}\mathfrak{S}_1^{\infty}\{F(\phi_{\alpha_1},\cdots,\phi_{\alpha_k},\psi_{\alpha_{k+1}},\cdots,\psi_{\alpha_n})\} = \lim_{m\to\infty} {}_{m}\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}\{F(\theta_1,\cdots,\theta_n)\}.$$

We shall now state some applications of the above formulas.

Application to partitions. We denote by $p_n(m)$ the number of partitions of m into parts not exceeding n or into at most n parts. Let $\{\beta_1\cdots\beta_n\}$ be an ordered partition of m, namely,

$$m=\beta_1+\beta_2+\cdots+\beta_n, \qquad \beta_1\geqq\beta_2\geqq\cdots\geqq\beta_n.$$

We further denote by $p_{a_1\cdots a_n}^{b_1\cdots b_n}(m)$ the number of ordered partitions, $\{\beta_1\cdots\beta_n\}$'s, of m into exactly n parts which are restricted to $a_1\leqq\beta_1\leqq b_1$, $a_2\leqq\beta_2\leqq b_2,\cdots,a_n\leqq\beta_n\leqq b_n$. We have then

$$(7)\quad p_{a_1\cdots a_n}^{b_1\cdots b_n}(m)=\sum_{k=0}^{n}(-1)^k\sum_{(\alpha_1\cdots\alpha_k)\in(1\cdots n)} p_n(m-k-b_{\alpha_1}-\cdots -b_{\alpha_k}-a_{\alpha_{k+1}}-\cdots-a_{\alpha_n}),$$

where the $p_n(m)$'s can be evaluated by the generating function

$$(8)\quad G_n(x)=\frac{1}{(1-x)(1-x^2)\cdots(1-x^n)}=1+\sum_{m=1}^{\infty}p_n(m)x^m.$$

PROOF OF (7). Starting with (5) we define

$$\theta_i(x)=\begin{cases}1 & \text{if } a_i\leqq x\leqq b_i,\\ 0 & \text{if } x<a_i \text{ or } b_i<x,\end{cases} \qquad (i=1\cdots n),$$

$$F(\theta_1(x_1),\cdots,\theta_n(x_n)) = \begin{cases}\theta_1(x_1)\cdots\theta_n(x_n) & \text{if the order } x_1\geqq x_2\geqq\cdots\geqq x_n \text{ holds,}\\ 0 & \text{if the order } x_1\geqq x_2\geqq\cdots\geqq x_n \text{ does not hold.}\end{cases}$$

Thus we see that

$$ {}_m\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}\{F(\theta_1, \cdots, \theta_n)\} = p_{a_1\cdots a_n}^{b_1\cdots b_n}(m). $$

Similarly

$$ {}_{m'}\mathfrak{S}_1^{\infty}\{F(\phi_{\alpha_1}, \cdots, \phi_{\alpha_k}, \psi_{\alpha_{k+1}}, \cdots, \psi_{\alpha_n})\} $$
$$ = p_n(m - b_{\alpha_1} - \cdots - b_{\alpha_k} - a_{\alpha_{k+1}} - \cdots - a_{\alpha_n} - k). $$

Hence (7) is proved.

It may be noted that the formula (7) still holds when $p_{a_1\cdots a_n}^{b_1\cdots b_n}(m)$ denotes the number of partitions of m into n parts which are restricted to more conditions than

$$ \beta_1 \geqq \beta_2 \geqq \cdots \geqq \beta_n; \quad a_1 \leqq \beta_1 \leqq b_1, \cdots, a_n \leqq \beta_n \leqq b_n. $$

Application to number of solutions. Let $A_{a_1\cdots a_n}^{b_1\cdots b_n}(N)$ denote the number of integral (or prime) solutions of the equation

$$ x_1^k + x_2^k + \cdots + x_n^k = N $$

with $a_1 \leqq x_1 < b_1$, $a_2 \leqq x_2 < b_2, \cdots, a_n \leqq x_n < b_n$, then by (5) we have

$$ (9) \quad A_{a_1\cdots a_n}^{b_1\cdots b_n}(N) = \sum_{k=0}^{n} (-1)^k \sum_{(\nu_1\cdots\nu_k)\in(1\cdots n)} A_{b_{\nu_1}\cdots b_{\nu_k} a_{\nu_{k+1}}\cdots a_{\nu_n}}^{\infty\cdots\infty\;\infty\cdots\infty}(N), $$

where $(\nu_1 \cdots \nu_k \cdots \nu_n) = (1 \cdots n)$.

We shall now proceed to find an asymptotic formula concerning the number of integral solutions of a linear equation with integral coefficients.

Let $A(N)$ denote the number of integral solutions of the equation

$$ \sum_{k=1}^{n} c_k x_k = N \qquad (c_1 \geqq 1, \cdots, c_n \geqq 1) $$

under the restrictions $\alpha_k N < x_k < \beta_k N$ $(k = 1, \cdots, n)$, where $c_1, \cdots, c_n$ are relatively prime to each other, α_k, β_k are real values.

When $\alpha_k = 0$, $\beta_k = 1$ $(k = 1, \cdots, n)$ it is well known that[3]

$$ (10) \qquad A(N) = N^{n-1}/c_1 \cdots c_n \cdot (n-1)! + O(N^{n-2}). $$

Define

$$ \theta_i(x) = \begin{cases} 1 & \text{if } x \text{ is divisible by } c_i \text{ and the inequality } \alpha_i c_i N < x < \beta_i c_i N \text{ holds,} \\ & \hfill (1 \leqq i \leqq n) \\ 0 & \text{if } x \text{ is not divisible by } c_i \text{ or the inequality does not hold.} \end{cases} $$

[3] The proof of (10) can be obtained easily by the method of partial fractions.

Thus by (4) and (10) we obtain the following consequence.[4]

Let $S_k=\sum_{(\nu_1\cdots\nu_k)\in(1\cdots n)}\phi_{\nu_1\cdots\nu_k}$, where

$$\phi_{\nu_1\cdots\nu_k}=\begin{cases}0 & \text{for } (1-\beta_{\nu_1}c_{\nu_1}-\cdots-\alpha_{\nu_n}c_{\nu_n})\leqq 0,\\ (1-\beta_{\nu_1}c_{\nu_1}-\cdots-\beta_{\nu_k}c_{\nu_k}-\alpha_{\nu_{k+1}}c_{\nu_{k+1}}-\cdots-\alpha_{\nu_n}c_{\nu_n})^{n-1} & \\ & \text{for } (1-\beta_{\nu_1}c_{\nu_1}-\cdots-\alpha_{\nu_n}c_{\nu_n})>0.\end{cases}$$

Then

$$\frac{A(N)}{N^{n-1}}=\frac{S_0-S_1+\cdots\pm S_n}{c_1\cdots c_n\cdot(n-1)!}+O\left(\frac{1}{N}\right). \tag{11}$$

Evidently (11) may be seen as a generalization of (10). If $c_1=\cdots=c_n=1$, $a_k\leqq x_k\leqq b_k$ $(k=1,\cdots,n)$, we can express $A(N)$ more precisely as[5]

$$A(N)=\sum_{k=0}^{n}(-1)^k\sum_{(\nu_1\cdots\nu_k)\in(1\cdots n)}C_{N'(k)+(a_{\nu_1}+\cdots+a_{\nu_k})-(b_{\nu_1}+\cdots+b_{\nu_k}),\,n-1} \tag{12}$$

where $N'(k)=N+n-k-1-(a_1+\cdots+a_n)$.

Application to Dirichlet's integral. The following theorem is well known.

Let

$$I_{(0)}=\iint_D\cdots\int x_1^{\alpha_1-1}x_2^{\alpha_2-1}\cdots x_n^{\alpha_n-1}\cdot f(x_1+x_2+\cdots+x_n)dx_1dx_2\cdots dx_n,$$

where the variables $x_1, x_2, \cdots, x_n$ are restricted to the region

$$D\colon\ 0\leqq k_1\leqq x_1+x_2+\cdots+x_n\leqq k_2;\ 0\leqq x_1, 0\leqq x_2,\cdots,0\leqq x_n.$$

Then the integral $I_{(0)}$ can be reduced to the form

$$I_{(0)}=\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_n)}\int_{k_1}^{k_2}u^{\alpha_1+\alpha_2+\cdots+\alpha_n-1}f(u)du. \tag{13}$$

We shall now extend the formula (13). Let

$$I\begin{Bmatrix}b_1,\cdots,b_n\\ x_1,\cdots,x_n\\ a_1,\cdots,a_n\end{Bmatrix}=\iint_R\cdots\int x_1^{\alpha_1-1}x_2^{\alpha_2-1}\cdots x_n^{\alpha_n-1}\cdot f(x_1+x_2+\cdots+x_n)dx_1dx_2\cdots dx_n,$$

[4] The detailed proof of this is omitted.

[5] The formula (12) can be obtained also by considering two generating functions.

where the region R is defined as the set of all points $(x_1 \cdots x_n)$ such that

$$0 \leqq a_i \leqq x_i \leqq b_i; \qquad x_1 + x_2 + \cdots + x_n \leqq k.$$

Since the Dirichlet integral $I_{(0)}$ can be considered as a limit of multiple summations with variable upper limits, by applying (6) we have

$$I \begin{Bmatrix} b_1 \cdots b_n \\ x_1 \cdots x_n \\ a_1 \cdots a_n \end{Bmatrix} = \sum_{j=0}^{n} (-1)^j \sum_{(\nu_1 \cdots \nu_j) \in (1 \cdots n)} I \begin{Bmatrix} \infty \cdots \infty & \infty \cdots \infty \\ x_{\nu_1} \cdots x_{\nu_j} & x_{\nu_{j+1}} \cdots x_{\nu_n} \\ b_{\nu_1} \cdots b_{\nu_j} & a_{\nu_{j+1}} \cdots a_{\nu_n} \end{Bmatrix}. \tag{14}$$

We have to establish a formula for

$$I \begin{Bmatrix} \infty \cdots \infty \\ x_1 \cdots x_n \\ c_1 \cdots c_n \end{Bmatrix} \qquad (c_1 \geqq 0, \cdots, c_n \geqq 0).$$

Let $I_{(1 \cdots s)}$ denote the integral

$$\frac{\Gamma(\alpha_{s+1}) \cdots \Gamma(\alpha_n)}{\Gamma(\alpha_{s+1} + \cdots + \alpha_n)} \int_0^{\min(c_1, k)} x_1^{\alpha_1 - 1} dx_1 \int_0^{\min(c_2, k - x_1)} x_2^{\alpha_2 - 1} dx_2 \cdots \int_0^{\min(c_s, k - x_1 - \cdots - x_{s-1})} x_s^{\alpha_s - 1} dx_s \int_0^{k - x_1 - \cdots - x_s} u^{\alpha_{s+1} + \cdots + \alpha_n - 1} f(u + x_1 + \cdots + x_s) du.$$

Now, by (14) it can be shown that

$$I \begin{Bmatrix} \infty \cdots \infty \\ x_1 \cdots x_n \\ c_1 \cdots c_n \end{Bmatrix} = I_{(0)} - \sum_{(i) \in (1 \cdots n)}^{C_{n,1}} I_{(i)} + \sum_{(ij) \in (1 \cdots n)}^{C_{n,2}} I_{(ij)} - \cdots + (-1)^n I_{(1 \cdots n)}, \tag{15}$$

where

$$I_{(0)} = I \begin{Bmatrix} \infty \cdots \infty \\ x_1 \cdots x_n \\ 0 \cdots 0 \end{Bmatrix},$$

$$I_{(1 \cdots n)} = \int_0^{\min(c_1, k)} \cdots \int_0^{\min(c_n, k - x_1 - \cdots - x_{n-1})} x_1^{\alpha_1 - 1} \cdots x_n^{\alpha_n - 1} dx_1 \cdots dx_n.$$

To prove (15), the formal logic theorem is also applicable.

Consider a differential $\bar{x}_1^{\alpha_1-1} \cdots \bar{x}_n^{\alpha_n-1} f(\bar{x}_1+\cdots+\bar{x}_n)d\bar{x}_1\cdots d\bar{x}_n$. We may assume that $\bar{x}_{\nu_1} \leqq c_{\nu_1}, \cdots, \bar{x}_{\nu_t} \leqq c_{\nu_t}, \bar{x}_{\nu_{t+1}} > c_{\nu_{t+1}}, \cdots, \bar{x}_{\nu_n} > c_{\nu_n}$, $(\nu_1 \cdots \nu_n) = (1 \cdots n)$.

Since the integral is a limit of multiple summations and may be written as

$$\begin{aligned} I_{(1\cdots s)} &= \int_0^{\min(c_1,k)} x_1^{\alpha_1-1} dx_1 \cdots \int_0^{\min(c_s, k-x_1-\cdots-x_{s-1})} x_s^{\alpha_s-1} dx_s \\ &\quad \cdot \int_{R_1} \cdots \int x_{s+1}^{\alpha_{s+1}-1} \cdots x_n^{\alpha_n-1} f(x_1+\cdots+x_n) dx_{s+1}\cdots dx_n \\ &= \int_0^{c_1}\cdots\int_0^{c_s}\int_{R_2(x_1\cdots x_n)}\cdots\int x_1^{\alpha_1-1}\cdots x_s^{\alpha_s-1} x_{s+1}^{\alpha_s+1} \cdots x_n^{\alpha_n} \\ &\qquad\qquad \cdot f(x_1+\cdots+x_n)dx_1\cdots dx_n, \end{aligned}$$

where

$$\begin{aligned} R_1(x_{s+1}\cdots x_n)\colon &\quad 0 \leqq x_{s+1}+\cdots+x_n \leqq k-(x_1+\cdots+x_s); \\ &\qquad\qquad x_{s+1},\cdots,x_n \geqq 0, \\ R_2(x_1\cdots x_n)\colon &\quad 0 \leqq x_1 \leqq c_1, \cdots, 0 \leqq x_s \leqq c_s; \\ &\quad 0 \leqq x_1+\cdots+x_n \leqq k; \qquad x_{s+1},\cdots,x_n \geqq 0, \end{aligned}$$

we see that the differential $\bar{x}_1^{\alpha_1-1} \cdots \bar{x}_n^{\alpha_n-1} f(\bar{x}_1+\cdots+\bar{x}_n)d\bar{x}_1\cdots d\bar{x}_n$ appears exactly $C_{t,s}$ times in $\sum_{(\nu_1\cdots\nu_s)\in(1\cdots n)} I_{(\nu_1\cdots\nu_s)}$. Therefore the number of occurrences of the given differential in the right-hand side of (15) is equal to

$$C_{t,0} - C_{t,1} + \cdots + (-1)^t C_{t,t} = \begin{cases} 1 & \text{if} \quad t = 0, \\ 0 & \text{if} \quad t > 0. \end{cases}$$

Hence the formula (15) is proved.

The integral $I_{(1\cdots s)}$ may be calculated by dividing the limits of the integral and integrating it separately.

It is seen that the integral $I_{(1\cdots s)}$ can be written also in the form:

$$\begin{aligned} &\frac{\Gamma(\alpha_{s+1})\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_{s+1}+\cdots+\alpha_n)} \int_0^{c_1} x_1^{\alpha_1-1}dx_1 \int_0^{c_2} x_2^{\alpha_2-1} dx_2 \cdots \int_0^{c_s} x_s^{\alpha_s-1} dx_s \\ &\cdot \int_0^{[(k-x_1-\cdots-x_s)+|k-x_1-\cdots-x_s|]/2} u^{\alpha_{s+1}+\cdots+\alpha_n-1} f(u+x_1+\cdots+x_s)du. \end{aligned}$$

Connecting (14) with (15), we see that it is a generalization of Dirichlet's integral $I_{(0)}$ with $k_2=0$, $k_1=k$ in (13).

It may be noted that the formula (14) is also called Liouville's extension and the integral regions D and R can be defined also by

$$D:\quad 0 \leqq k_1 \leqq a_1 x_1^{p1} + \cdots + a_n x^{p_n} \leqq k_2, \qquad 0 \leqq x_i;$$
$$R:\quad 0 \leqq a_i \leqq x_i \leqq b_i, \qquad d_1 x_1^{q1} + \cdots + d x_n^{q_n} \leqq k.$$

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TRANSFORMATIONS IN METRIC SPACES AND ORDINARY DIFFERENTIAL EQUATIONS

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1. Introduction. It is evident that the solutions of a differential equation $y' = f(t, y)$ passing through a point (τ, η) in the region of definition of $f(t, y)$ may be considered as invariant functions of the transformation $Ty(t) = \eta + \int_\tau^t f(s, y(s))ds$ when suitable restrictions are placed upon the functions $y(t)$ considered. That such invariant functions exist for continuous $f(t, y)$ can be made a consequence of Schauder's fixed point theorem for completely continuous transformations in bounded convex subsets of a Banach space.[1] For $f(t, y)$ satisfying a Lipschitz condition in y the existence and uniqueness of an invariant function can be made to follow from a fixed point theorem of Caccioppoli of an essentially simpler nature.[2] In the present paper we wish to show that the existence of invariant functions for continuous $f(t, y)$ as well as several other theorems concerning solutions of differential equations can be made to follow from some theorems concerning a particular class of transformations in a complete metric space. Although the existence theorem for fixed points given

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[1] J. Schauder, *Der Fixpunktsatz in Funktionalräumen*, Studia Mathematica vol. 2 (1930) pp. 171–180; also, *Zur Theorie stetiger Abbildungen in Funktionalräumen*, Math. Zeit. vol. 26 (1927) pp. 47–65 and 417–431.

[2] R. Caccioppoli, *Un teorema generale sull'esistenza di elementi uniti in una trasformazione funzionale*, Rendiconti R. Accademia dei Lincei (6) vol. 11 (1930) pp. 794–799; for proofs and various applications of both Caccioppoli's and Schauder's theorems we refer also to Niemytsky, *Metod nepodvizhnykh Tochek v Analize*, Uspekhi Mat. Nauk vol. 1 (1936) pp. 141–174.