It may be noted that the formula (14) is also called Liouville's extension and the integral regions D and R can be defined also by

$$D: \quad 0 \leqq k_1 \leqq a_1 x_1^{p_1} + \cdots + a_n x^{p_n} \leqq k_2, \qquad 0 \leqq x_i;$$
$$R: \quad 0 \leqq a_i \leqq x_i \leqq b_i, \qquad d_1 x_1^{q_1} + \cdots + d x_n^{q_n} \leqq k.$$

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TRANSFORMATIONS IN METRIC SPACES AND ORDINARY DIFFERENTIAL EQUATIONS

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1. Introduction. It is evident that the solutions of a differential equation $y' = f(t, y)$ passing through a point (τ, η) in the region of definition of $f(t, y)$ may be considered as invariant functions of the transformation $Ty(t) = \eta + \int_\tau^t f(s, y(s))ds$ when suitable restrictions are placed upon the functions $y(t)$ considered. That such invariant functions exist for continuous $f(t, y)$ can be made a consequence of Schauder's fixed point theorem for completely continuous transformations in bounded convex subsets of a Banach space.[1] For $f(t, y)$ satisfying a Lipschitz condition in y the existence and uniqueness of an invariant function can be made to follow from a fixed point theorem of Caccioppoli of an essentially simpler nature.[2] In the present paper we wish to show that the existence of invariant functions for continuous $f(t, y)$ as well as several other theorems concerning solutions of differential equations can be made to follow from some theorems concerning a particular class of transformations in a complete metric space. Although the existence theorem for fixed points given

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[1] J. Schauder, *Der Fixpunktsatz in Funktionalräumen*, Studia Mathematica vol. 2 (1930) pp. 171–180; also, *Zur Theorie stetiger Abbildungen in Funktionalräumen*, Math. Zeit. vol. 26 (1927) pp. 47–65 and 417–431.

[2] R. Caccioppoli, *Un teorema generale sull'esistenza di elementi uniti in una trasformazione funzionale*, Rendiconti R. Accademia dei Lincei (6) vol. 11 (1930) pp. 794–799; for proofs and various applications of both Caccioppoli's and Schauder's theorems we refer also to Niemytsky, *Metod nepodvizhnykh Tochek v Analize*, Uspekhi Mat. Nauk vol. 1 (1936) pp. 141–174.

here is much more elementary than Schauder's theorem, it is no more difficult to apply to ordinary differential equations provided one is willing to admit that differential equations for which $f(t, y)$ is a polynomial in t and y have unique solutions.

2. **Transformations in metric spaces.** Let M be a complete metric space. A transformation T of M into itself will be called completely continuous if (1) it is continuous, (2) $T(M)$ is bounded, and (3) the image of any bounded set in M has a compact closure. If M is bounded, condition (2) is of course trivial. The "distance" between two such transformations may be defined by $\rho^*(T_1, T_2) = \sup_{x \in M} \rho(T_1x, T_2x)$; this is clearly finite from condition (2). If $\mathfrak{T}$ is any class of such transformations it is easy to see that this distance function may be used to convert $\mathfrak{T}$ into a metric space. It is a routine matter to prove the following theorem.

THEOREM. *If $\mathfrak{T}$ is the class of all completely continuous transformations defined on M and metrized as above, then $\mathfrak{T}$ is complete.*

In general, however, $\mathfrak{T}$ will not denote the class of all completely continuous transformations but merely some subclass. Moreover, it will be advantageous to alter the distance function in $\mathfrak{T}$ by substituting for $\rho^*(T_1, T_2)$ any distance function $\rho(T_1, T_2)$ which is at least as "strong" as $\rho^*(T_1, T_2)$, that is, one such that $\lim \rho(T_n, T) = 0$ implies $\lim \rho^*(T_n, T) = 0$. Topological properties of subsets of $\mathfrak{T}$ such as "closed," "dense," and so on are to be understood in terms of the topology defined by the function $\rho(T_1, T_2)$. There seems to be little danger of confusing this function ρ with that defined in M.

Let $\mathfrak{T}_0$ denote those elements of $\mathfrak{T}$ having no fixed points in M, $\mathfrak{T}_1$ those elements having exactly one fixed point in M, and $\mathfrak{T}_2$ those having more than one fixed point. For a given $T \in \mathfrak{T}$ let $F(T)$ denote the set of fixed points of T. This set is clearly closed and compact. For any $x \in M$, $\epsilon > 0$, let $S(x, \epsilon) = E_{y \in M}[\rho(x, y) < \epsilon]$ and for any set $A \subset M$ let $U(A, \epsilon)$ be the union of all $S(x, \epsilon)$ for $x \in A$; analogous notation will be used for elements and subsets of $\mathfrak{T}$.

THEOREM 1. *There exists $\delta(T) > 0$ defined for $T \in \mathfrak{T}_0$ such that $\rho(T', T) < \delta(T)$ implies $T' \in \mathfrak{T}_0$, that is, $\mathfrak{T}_0$ is open in $\mathfrak{T}$.*

For suppose there exist $T_n \in \mathfrak{T}_1 + \mathfrak{T}_2$ and $T \in \mathfrak{T}_0$ such that $\rho(T_n, T) < 1/n$. Let $x_n \in F(T_n)$. It is easily seen that the sequence $\{x_n\}$ is bounded. Since T is completely continuous there exist x_{n_i}, x_0 such that $\rho(Tx_{n_i}, x_0) \to 0$. But $\rho(x_{n_i}, x_0) \leqq \rho(T_{n_i}x_{n_i}, Tx_{n_i}) + \rho(Tx_{n_i}, x_0)$ $\leqq \rho^*(T_{n_i}, T) + \rho(Tx_{n_i}, x_0)$ which approaches 0 as $i \to \infty$. But then, since T is continuous, $\rho(Tx_{n_i}, Tx_0)\to 0$ which implies that $Tx_0=x_0$, contradicting the choice of $T\in\mathfrak{T}_0$.

THEOREM 2. *There exists* $\delta(T, \epsilon)>0$ *defined for* $T\in\mathfrak{T}_1+\mathfrak{T}_2$ *and* $\epsilon>0$ *such that* $\rho(T', T)<\delta(T, \epsilon)$ *implies* $F(T')\subset U(F(T), \epsilon)$.

If $F(T')$ is empty the conclusion is trivially true. Suppose that for some $\epsilon>0$ and $T\in\mathfrak{T}_1+\mathfrak{T}_2$ there exist $T_n\in\mathfrak{T}_1+\mathfrak{T}_2$ and $x_n\in F(T_n)$ such that $\rho(T_n, T)<1/n$ but $x_n\notin U(F(T), \epsilon)$. Since T is completely continuous there exist x_{n_i}, x_0 such that $\rho(Tx_{n_i}, x_0)\to 0$. It now follows as in the previous theorem that $\rho(x_{n_i}, x_0)\to 0$ and $Tx_0=x_0$, that is, $x_0\in F(T)$ which contradicts the assumption that $x_n\notin U(F(T), \epsilon)$.

THEOREM 3. *If* $\mathfrak{T}_1+\mathfrak{T}_2$ *is dense in* $\mathfrak{T}$, $\mathfrak{T}_0$ *is empty.*

This is an immediate consequence of Theorem 1 since the complement of a dense set cannot be open unless it is empty.

THEOREM 4. *If* $\mathfrak{T}_1$ *is dense in* $\mathfrak{T}$, $\mathfrak{T}_0$ *is empty and* $\mathfrak{T}_2$ *is of the first category in* $\mathfrak{T}$. *If* $\mathfrak{T}$ *is complete,* $\mathfrak{T}_1$ *is of the second category in* $\mathfrak{T}$.

That $\mathfrak{T}_0$ is empty follows from Theorem 3. Let $d[A]$ denote the diameter of $A\subset M$. Then $T\in\mathfrak{T}_2$ is equivalent to $d[F(T)]\geqq 0$. Clearly, $\mathfrak{T}_2=\mathfrak{N}_1+\mathfrak{N}_2+\cdots+\mathfrak{N}_i+\cdots$ where $\mathfrak{N}_i=E[T\in\mathfrak{T}, d[F(T)]\geqq 1/i]$. If it is shown that $\mathfrak{N}_i$ is nowhere-dense, the statement concerning $\mathfrak{T}_2$ will be proved. Let $T_0\in\mathfrak{T}$, $\epsilon>0$ be arbitrary. Since $\mathfrak{T}_1$ is dense in $\mathfrak{T}$, there exists $T_1\in\mathfrak{T}_1$ such that $T_1\in S(T_0, \epsilon/2)$. But also there exists $\eta<\epsilon/2$ such that $S(T_1, \eta)\subset S(T_0, \epsilon)$ and such that $S(T_1, \eta)$ has an empty intersection with $\mathfrak{N}_i$. For, by Theorem 2, if $\eta<\min\,[\epsilon/2, \delta(T_1, 1/3i)]$ and $T\in S(T_1, \eta)$, then $F(T)\subset U(F(T_1), 1/3i)$ and consequently $d[F(T)]\leqq d[F(T_1)]+2/3i<1/i$, that is, $T\notin\mathfrak{N}_i$. This is just the condition that $\mathfrak{N}_i$ be nowhere-dense. The final statement follows from the facts that $\mathfrak{T}=\mathfrak{T}_1+\mathfrak{T}_2$ and that $\mathfrak{T}$, being complete, is of the second category in itself.

THEOREM 5. *If* $\mathfrak{T}_2$ *is dense in* $\mathfrak{T}$ *and* $\mathfrak{A}\subset\mathfrak{T}_1$ *is closed in* $\mathfrak{T}$, *then* $\mathfrak{A}$ *is nowhere-dense in* $\mathfrak{T}$.

For suppose $\mathfrak{A}$ is not nowhere-dense in $\mathfrak{T}$. Then there is some sphere in $\mathfrak{T}$ in which $\mathfrak{A}$ is dense and which consequently $\mathfrak{A}$ contains since it is closed. But this contradicts the hypothesis concerning $\mathfrak{T}_2$.

3. **Applications.** We wish now to apply these theorems to systems of real differential equations of the form $y_i'=f_i(t, y_1, \cdots, y_n)$, $i=1, \cdots, n$. In order to do this we shall introduce certain metric spaces, in this case Banach spaces. Let R be the set of points

$(t, y_1, \cdots, y_n)$ in E_{n+1} determined by the inequalities $a \leqq t \leqq b$ and $c_i \leqq y_i \leqq d_i$, $i=1, \cdots, n$, where $a<b$, $c_i<d_i$ and any of the values c_i, d_i may be infinite. Let $C_n(a, b)$ be the Banach space of n-tuples of functions $Y(t)=(y_1(t), \cdots, y_n(t))$, each $y_i(t)$ being continuous on $[a, b]$, and with norm defined by $\|Y\|=\sum_{i=1}^n\|y_i\|=\sum_{i=1}^n \max_{a\leqq t\leqq b}|y_i(t)|$. Let $C_n(R)$ be the Banach space of n-tuples of functions $F(t, y_1, \cdots, y_n) =(f_1(t, y_1, \cdots, y_n), \cdots, f_n(t, y_1, \cdots, y_n))$, each f_i being continuous and bounded on R, with norm $\|F\|=\sum_{i=1}^n\|f_i\|=\sum_{i=1}^n \sup_R|f_i(t, y_1, \cdots, y_n)|$. Let M be the subset of $C_n(a, b)$ consisting of elements Y such that $c_i \leqq y_i(t) \leqq d_i$. M is a complete metric space.

For any $P(\tau, \eta_1, \cdots, \eta_n) \in R$ and $F\in C_n(R)$ we may define the following transformation in M: $TY=(T_1Y, \cdots, T_nY)$ where for $a\leqq t\leqq b$

$$T_iY(t) = \begin{cases} d_i & \text{if } \eta_i + \int_\tau^t f_i ds > d_i, \\ \eta_i + \int_\tau^t f_i(s, y_1(s), \cdots, y_n(s))ds & \text{if } c_i \leqq \eta_i + \int_\tau^t f_i ds \leqq d_i, \\ c_i & \text{if } c_i > \eta_i + \int_\tau^t f_i ds. \end{cases}$$

Each transformation T is thus associated with a definite point $P\in R$ and a definite $F\in C_n(R)$. Clearly, any fixed point $Y(t)$ of this transformation gives a solution in R through the point P of the set of differential equations $y_i' = f_i(t, y_1, \cdots, y_n)$, $i=1, \cdots, n$, after one has taken proper account of the modification of TY on the boundary of R. The actual solution of the differential equations will be a subcontinuum containing P of the curve representing $Y(t)$. This solution will not be a "local solution" but will extend in both directions[3] to the boundary of R. At the left and right end points of the curve representing the actual solution y_i' will of course be interpreted as the right and left derivative respectively.

The class $\mathfrak{T}$ which we shall consider consists of all transformations of the form above associated with some $P\in R$ and $F\in C_n(R)$. A metric in $\mathfrak{T}$ is defined by $\rho(T', T'')=|\tau'-\tau''|+\sum_{i=1}^n|\eta_i'-\eta_i''|+\|F'-F''\|$. Since R and $C_n(R)$ are complete $\mathfrak{T}$ is a complete metric space. It follows from the form of T that $TM\subset M$. That TM is bounded follows from the boundedness on R of the functions f_i. Continuity of T

[3] It may happen, however, in case some of the values η_i are equal to c_i or d_i, that the curve representing the actual solution will reduce to the single point P. This will not happen if P is interior to R.

follows from the uniform continuity of the functions f_i on bounded closed subsets of R. That T takes bounded subsets of M into sets with compact closures follows from the equicontinuity of the functions $T_iY(t)$ for $Y \in M$ and Arzela's criterion for compactness in the space $C_n(a, b)$. One may conclude from the following inequalities that $\rho(T', T'')$ is at least as strong as $\rho^*(T', T'')$: $\rho^*(T', T'') = \sum_{i=1}^{n} \sup_{Y\in M} \|T_i' Y - T_i'' Y\| \leqq \sum_{i=1}^{n} \sup_{Y\in M} \max_{a\leqq t\leqq b} \left| \eta_i' - \eta_i'' + \int_{\tau'}^{t} f_i'(s, y_1(s), \cdots, y_n(s))ds - \int_{\tau''}^{t} f_i''(s, y_1(s), \cdots, y_n(s))ds \right| \leqq \sum_{i=1}^{n} | \eta_i' - \eta_i'' | + \sum_{i=1}^{n} \sup_{Y\in M} \left| \int_{\tau'}^{\tau''} f_i' ds \right| + \sum_{i=1}^{n} \sup_{Y\in M} \int_a^b | f_i' - f_i''| ds$, which evidently becomes small as $\rho(T', T'')$ becomes small.

In order to apply Theorem 4 to $\mathfrak{T}$ we must still show that $\mathfrak{T}_1$ is dense in $\mathfrak{T}$. But this follows from the well known facts that the elements F whose components are polynomials are dense in $C_n(R)$ and that for such $F \in C_n(R)$ the corresponding set of differential equations $y_i' = f_i(t, y_1, \cdots, y_n)$, $i = 1, \cdots, n$, has a unique solution through every $P \in R$. From Theorem 4 we may now conclude that $\mathfrak{T}_0$ is empty and, consequently, that every set of differential equations of the kind considered has at least one solution through every point in R. Theorem 4 gives the additional information that those sets of differential equations, associated with some $P \in R$ and $F \in C_n(R)$, which have multiple solutions are of the first category in $\mathfrak{T}$ whereas those with unique solutions are of the second category. This theorem is closely related to the following theorem of Orlicz:[4] Let G be a domain in E_2 and $C(G)$ the Banach space of bounded continuous functions $f(t, y)$ defined on G. Then those functions $f(t, y)$ for which $y' = f(t, y)$ has a unique solution through every point of G are of the second category in $C(G)$ whereas the complementary set is of the first category. The chief difference between the theorems seems to be that Orlicz's theorem is concerned with a property of the element F while the theorem above deals with a property of the pair (P, F). Orlicz's theorem does not seem to follow readily from Theorem 4 but requires additional discussion.

Theorem 2, when applied to our special choice of M and $\mathfrak{T}$, gives the following result. If $P_k(\tau_k, \eta_{1k}, \cdots, \eta_{nk}) \rightarrow P(\tau, \eta_1, \cdots, \eta_n)$ where $c_i < \eta_i < d_i$ and $\|F_k - F\| \rightarrow 0$, then there exists an interval $[a', b']$ with $a \leqq a' \leqq \tau \leqq b' \leqq b$ and $a' < b'$ such that for any $\epsilon > 0$ there exists k_ϵ such that for $k > k_\epsilon$ each solution through P_k of $y_i' = f_{ik}(t, y_1, \cdots, y_n)$, $i = 1, \cdots, n$, is within norm distance ϵ of some solution through P of $y_i' = f_i(t, y_1, \cdots, y_n)$, $i = 1, \cdots, n$, when one restricts consideration of the solutions to $[a', b']$. In case $y_i' = f_i(t, y_1, \cdots, y_n)$,

[4] W. Orlicz, *Zur Theorie der Differentialgleichung $y' = f(x, y)$*, Bulletin International de l'Académie Polonaise des Sciences et Lettres, Série A, Nos. 8, 9 (1932) pp. 221–228.

$i=1, \cdots, n$, has a unique solution through P, the solutions through P_k of $y_i' = f_{ik}(t, y_1, \cdots, y_n)$, $i=1, \cdots, n$, will converge uniformly on $[a', b']$ to this solution.[5] It is evident that the indices k do not need to be integers but could be elements of any directed set, for example, a set of r parameters upon which the point P and the functions F depend continuously in the sense of our topology. That it may be necessary to consider the solutions only on some subinterval of $[a, b]$ is clear when one recalls the modification of the transformation T on the boundary of R.

In order to apply Theorem 5 we must first show that $\mathfrak{T}_2$ is dense in $\mathfrak{T}$. Let $P(\tau, \eta_1, \cdots, \eta_n)$ and $F \in C_n(R)$ be the point and functions corresponding to an arbitrary transformation $T \in \mathfrak{T}$. For any $\epsilon > 0$ select $\alpha > 0$ and $\beta_i > 0$ so that in the neighborhood $|t-\tau| < \alpha$, $|y_i - \eta_i| < \beta_i$ one has $|f_i(t, y_1, \cdots, y_n) - f_i(\tau, \eta_1, \cdots, \eta_n)| < \epsilon/2$, $i=1, \cdots, n$. In this neighborhood define $g_i(t, y_1, \cdots, y_n) = f_i(\tau, \eta_1, \cdots, \eta_n) + \gamma |y_i - \eta_i - (t-\tau) f_i(\tau, \eta_1, \cdots, \eta_n)|^{1/2}$ where γ is chosen so small that in the neighborhood one has $|g_i - f_i| < \epsilon$. Then extend g_i over R as a continuous function in such a manner that $|f_i - g_i| < \epsilon$ on R. The transformation T' associated with P and G will then be such that $\rho(T, T') < \epsilon$. But $T' \in \mathfrak{T}_2$, for the set of equations $y_i' = g_i(t, y_1, \cdots, y_n)$, $i=1, \cdots, n$, has more than one solution (in fact, continuum-many) through the point P. In the neighborhood defined above two of them are: $y_i(t) = \eta_i + (t-\tau) f_i(\tau, \eta_1, \cdots, \eta_n)$, $i=1, \cdots, n$, and $y_i(t) = \eta_i + (t-\tau) f_i(\tau, \eta_1, \cdots, \eta_n) + (\gamma/2)^2 (t-\tau)^2 sgn(t-\tau)$, $i=1, \cdots, n$. These functions will then extend over R to two different solutions.[6] Since $\epsilon > 0$ was arbitrary, $\mathfrak{T}_2$ is dense in $\mathfrak{T}$. To see the significance of Theorem 5 we remark that the set of $F \in C_n(R)$ satisfying a uniform Lipschitz condition on R with a fixed constant $m > 0$, that is, those $(f_1, \cdots, f_n)$ with $|f_i(t, \bar{y}_1, \cdots, \bar{y}_n) - f_i(t, y_1, \cdots, y_n)| \leqq m \cdot \sum_{i=1}^{n} |\bar{y}_i - y_i|$ on R, is closed in $C_n(R)$. Consequently, the set $\mathfrak{A}(m)$ of transformations associated with an arbitrary $P \in R$ and any F in the class above is closed in $\mathfrak{T}$. Moreover, as is well known, $\mathfrak{A}(m) \subset \mathfrak{T}_1$. But then, from Theorem 5, $\mathfrak{A}(m)$ is nowhere-dense in $\mathfrak{T}$. Since the class of transformations with the associated F satisfying a uniform Lipschitz condition on R for any $m > 0$ is given by $\sum_{m=1}^{\infty} \mathfrak{A}(m)$, this class is of the first category in $\mathfrak{T}$. This theorem can be weakened to the following: The class $\mathfrak{B}$ of transformations such that F satisfies a uniform Lipschitz condition in some neighborhood containing P is of the first

[5] This theorem is given by Kamke, *Differentialgleichungen reeler Funktionen*, Leipzig, 1930, p. 149, Theorems 3 and 4.

[6] Kamke, loc. cit. p. 135, Theorem 2.

category in $\mathfrak{T}$. This does not follow directly from Theorem 5 for this class is not contained in $\mathfrak{T}_1$. However, $\mathfrak{B}=\sum_{m=1}^{\infty}\sum_{r=1}^{\infty}\mathfrak{B}(m, r)$ where $\mathfrak{B}(m, r)$ is the class of transformations for which F satisfies a uniform Lipschitz condition with constant m in a neighborhood of radius $1/r$ about the point P associated with T. The class $\mathfrak{B}(m, r)$ may be shown to be closed. Inspection of the functions $(g_1, \cdots, g_n)$ defined above shows that the complement of $\mathfrak{B}(m, r)$ is dense in $\mathfrak{T}$ and, consequently, that $\mathfrak{B}(m, r)$ is nowhere-dense in $\mathfrak{T}$, and $\mathfrak{B}$ of the first category in $\mathfrak{T}$. Not only for uniform Lipschitz conditions, but also for more general uniqueness criteria is it true that the class of functions satisfying them is closed. Without going into details we mention the criterion given by Kamke[7] which, when satisfied at every point of R, gives a class of functions closed in $C_n(R)$ and consequently a subclass of $\mathfrak{T}_1$ nowhere-dense in $\mathfrak{T}$. These theorems are also closely related to theorems of Orlicz[8] where again the present theorems differ from Orlicz's in the way described above.

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[7] Kamke, loc. cit. p. 139, Theorem 3.
[8] Orlicz, loc. cit. pp. 226–228.