NONCOMMUTATIVE VALUATIONS

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The topic of this paper is the extension of the basic facts of valuation theory to noncommutative systems.[1] The purpose of this generalization is twofold. First, the theory of valuations with commutative groups of values is placed in the framework of the theory of l-groups,[2] and secondly the general theory leads to the construction of a new class of infinite division algebras. These division algebras are of highly transcendental structure over their respective centers; moreover they may be considered, in special cases, as crossed transcendental extensions of other division algebras.

It is necessary to recall some facts on l-groups. A group Γ is called a simply ordered l-group if the following axioms are satisfied:

(I) There is defined a binary inclusion relation which is "homogeneous" in the sense that $\alpha \geqq \beta$ implies $\rho+\alpha+\sigma \geqq \beta+\beta+\sigma$ for all ρ, σ,

(II) Γ is a lattice with respect to the ordering relation, and

(III) given α, β, either $\alpha \geqq \beta$ or $\beta \geqq \alpha$.[3]

Furthermore $\alpha \geqq \beta$ means $\alpha \cup \beta = \alpha$. The totality of all positive elements of Γ is a semi-group and shall be denoted by Γ^+. The absolute value $|\alpha|$ of α is defined as $\alpha \cup -\alpha$. Hence $|\alpha|$ is equal to α or $-\alpha$ according as α lies in Γ^+ or the complement $\Gamma - \Gamma^+$.[4] Since Γ is simply ordered an l-ideal or isolated subgroup Δ which is defined by Birkhoff[5] to contain with each δ all ξ with $|\xi| < |\delta|$ may alternately be defined as follows. An isolated subgroup contains with each $\delta > 0$ all $\xi \in \Gamma^+$ satisfying $\xi < \delta$.

DEFINITION. *A one-valued function V on a division ring D upon an l-group Γ is called a valuation if the following postulates hold:*

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[1] For results on valuation theory see, for example, W. Krull, *Allgemeine Bewertungstheorie*, J. Reine Angew. Math. vol. 167 (1932) pp. 160–196; O. F. G. Schilling, *Arithmetic in fields of formal power series in several variables*, Ann. of Math. vol. 38 (1937) pp. 551–576; Saunders MacLane, *The uniqueness of the power series representation of certain fields with valuations*, Ann. of Math. vol. 39 (1938) pp. 370–382; A. Ostrowski, *Untersuchungen zur arithmetischen Theorie der Körper*, Math. Zeit. vol. 39 (1935) pp. 269–404; I. Kaplansky, *Maximal fields with valuations*, Duke Math. J. vol. 9 (1942) pp. 303–321.

[2] Garrett Birkhoff, *Lattice-ordered groups*, Ann. of Math. vol. 43 (1942) pp. 298–331.

[3] Birkhoff, loc. cit. pp. 299, 300, 312.

[4] Birkhoff, loc. cit. pp. 302, 308, 309.

[5] Birkhoff, loc. cit. pp. 309–311.

(i) *Each* $\alpha \in \Gamma$ *has the form* Va *for at least one element* $a \in D$.
(ii) $V(0) > \alpha$ *for all* $\alpha \in \Gamma^+$.
(iii) $V(ab) = V(a) + V(b)$.
(iv) $V(a+b) \geqq \min [V(a), V(b)]$.

REMARK 1. If Γ is noncommutative then D is certainly not a field. This follows, by contradiction, from (i) and (iii).

REMARK 2. All elements u of D with $V(u)=0$ form a normal subgroup U^* of D for which $D^*/U \cong \Gamma$. This holds since V is a homomorphism of D^* upon Γ.

As in the ordinary valuation theory it is now shown, using (iii) and (iv), that the totality $\mathfrak{O}$ of all $a \in D$ with $Va \geqq 0$ is a ring, the valuation ring of V.

LEMMA 1. *For* a, b *in* $\mathfrak{O}$, *the following statements are equivalent*:
(i) $a = c_1 b$ *with* c_1 *in* $\mathfrak{O}$.
(ii) $a = bc_2$ *with* c_2 *in* $\mathfrak{O}$.
(iii) $V(a) \geqq V(b)$.

PROOF. Application of V to (i) or (ii) yields (iii). Conversely (iii) implies the existence of elements γ_1 and γ_2 in Γ^+ such that $V(a) = \gamma_1 + V(b)$ and $V(a) = V(b) + \gamma_2$. Then there exist elements d_1 and d_2 with $V(d_1) = \gamma_1$ and $V(d_2) = \gamma_2$. Hence $a = u_1 d_1 b = b d_2 u_2$ where $u_1 d_1$ and $d_2 u_2$ lie in $\mathfrak{O}$, u_1 and u_2 in U.

LEMMA 2. *Each ideal* $\mathfrak{A}$ *of* $\mathfrak{O}$ *is two-sided.*

PROOF. Suppose that $\mathfrak{A}$ is a left ideal, that is, $\mathfrak{O}\mathfrak{A} \subseteq \mathfrak{A}$. Let $V\mathfrak{A}$ be the set of all $V(a)$ with $a \in \mathfrak{A}$. This set is an upper class of Γ^+ as follows by Lemma 1. Now let $d = \sum_{j=1}^{N} a_j b_j$ be an arbitrary element of the set $\mathfrak{A}\mathfrak{O}$; $a_j \in \mathfrak{A}$, $b_j \in \mathfrak{O}$. Then $a_j b_j = b_j' a_j$, $b_j' \in \mathfrak{O}$ by Lemma 1. Consequently $d \in \mathfrak{A}$. Thus $\mathfrak{A}$ is a right ideal.

Now let $\mathfrak{P}$ be the ideal of elements $b \in \mathfrak{O}$ with $Vb > 0$. Moreover, $\mathfrak{O}/\mathfrak{P}$ is a division ring $\boldsymbol{D}$ for the elements of $\boldsymbol{D} - 0 = \boldsymbol{D}^*$ can be represented by elements in the multiplicative group U. As usual $\mathfrak{P}$ is termed the prime ideal of V.

LEMMA 3. *The sets* $\mathfrak{O}$ *and* $\mathfrak{P}$ *are invariants for the group of inner automorphisms of* D.

PROOF. Let $d \in D^*$ and $a \in \mathfrak{O}$. Then $V(d^{-1}ad) = -V(d) + V(a) + V(d) \geqq 0$ by the invariance of Γ^+. Hence $d^{-1}ad \in \mathfrak{O}$. Next one finds $V(d^{-1}ad) > 0$ for $a \in \mathfrak{P}$. The possibility $V(d^{-1}ad) = 0$ is excluded for $-V(d) + V(a) + V(d) = 0$ would imply $V(a) + V(d) = V(d)$, $V(a) = 0$, in contradiction to the assumption on a.

LEMMA 4. *Let* $a, b \in D^*$, *then at least one of the pairs* $\{ab^{-1}, b^{-1}a\}$, $\{a^{-1}b, ba^{-1}\}$ *lies in* $\mathfrak{D}$.

PROOF. Suppose that $ab^{-1} \in \mathfrak{D}$. Then also $b^{-1}a \in \mathfrak{D}$ by Lemma 1. Now assume $ab^{-1} \notin \mathfrak{D}$. Then, by the definition of $\mathfrak{D}$, $0 > V(ab^{-1})$. Consequently $-V(a)+V(b) > 0$ and thus $a^{-1}b$ and ba^{-1} lie in $\mathfrak{D}$ by Lemma 1. Moreover, $b^{-1}a \notin \mathfrak{D}$ for otherwise $-V(b)+V(a) \geqq 0$ or $V(a)-V(b) \geqq 0$ contrary to the assumption on the elements a and b.

REMARK. If $d \in D$ then either $d \in \mathfrak{D}$ or $d = a^{-1}$ where $a \in \mathfrak{D}$. For the latter observe that $0 > V(d)$ implies $-V(d) > V(d^{-1}) + V(d) = V(1) = 0$.

LEMMA 5. *If* $\mathfrak{D}$ *is an invariant subring of* D *such that for any* $a \in D$, *either* a *or* a^{-1} *is in* $\mathfrak{D}$, *then* $\mathfrak{D}$ *is a valuation ring for some valuation of* D.

PROOF. The invariance of the ring $\mathfrak{D}$ implies the invariance of its group of units U. For if $u \in U$, then $d^{-1}ud$ and $d^{-1}u^{-1}d = (d^{-1}ud)^{-1}$ both lie in $\mathfrak{D}$ for every $d \in D$, that is, $d^{-1}ud \in U$. Now let $\mathfrak{P}$ be the complement $\mathfrak{D} - U$. The set $\mathfrak{P}$ is invariant under the group of inner transformations of D. To define the valuation V for which $\mathfrak{D}$ is the valuation ring set $V(a) = aU$ for $a \in D^*$ and $V(u) = 0$ for $u \in U$. The factor group D^*/U may then be considered as an additive group Γ. The group turns out to be a simply ordered l-group, if $\alpha = V(a) > \beta = V(b)$ in case ab^{-1} and $b^{-1}a$ lie in $\mathfrak{P}$.[6] As in the commutative theory it now follows that $\mathfrak{D}$ is the valuation ring for V.[7]

The preceding properties of the valuation V lead to another description of a valuation. By the homomorphism $a \rightarrow a \bmod \mathfrak{P} = H(a) = \boldsymbol{a} \in \boldsymbol{D}$ exactly the valuation ring $\mathfrak{D}$ is mapped upon $\boldsymbol{D}$. It is customary to agree that $H(d) = \infty$ if $d \notin \mathfrak{D}$. This can happen only if $H(d^{-1}) = 0$, for $d \notin \mathfrak{D}$ means $d = a^{-1}$ with $V(a) > 0$, that is, $a \in \mathfrak{P}$, and thus $H(a) = 0$. Finally $H(d^{-1}ad) \neq \infty$ if and only if $H(a) \neq \infty$ for inner automorphisms preserve non-negativeness.

THEOREM 1. *Let* H *be a homomorphism of a division ring* D *upon a division ring* $\boldsymbol{D}$ *and a symbol* ∞ *so that* (i) $H(a+b) = H(a) + H(b)$ *and* $H(ab) = H(a)H(b)$ *for any pair* $a, b \in D$ *with* $H(a) \neq \infty$, $H(b) \neq \infty$, (ii) $H(a) = \infty$ *if and only if* $H(a^{-1}) = 0$, *and* (iii) $H(d^{-1}ad) \neq \infty$ *for all* $d \in D^*$ *if and only if* $H(a) \neq \infty$. *Then* H *arises from a valuation* V *of* D.

PROOF. Denote by $H^{-1}(\boldsymbol{S})$ the inverse image of a subset $\boldsymbol{S} \subseteq \boldsymbol{D}$. Then $H^{-1}(\boldsymbol{D}) = \mathfrak{D}$ is a ring with $H^{-1}(\boldsymbol{D} - 0) = U$ for its subgroup of units. Certainly $\mathfrak{D}$ and U are invariant under the group of inner

[6] Observe, for example, that $V(a) \geqq 0$ if and only if $a \in \mathfrak{D}$, $d^{-1}ad \in \mathfrak{D}$ for $d \in D^*$ implies that Γ is homogeneous.

[7] Observe that for any a, b in D^* either $a^{-1}b$ or $b^{-1}a$ lies in $\mathfrak{D}$. Say the former holds, then $a(a^{-1}b)a^{-1} = ba^{-1}$ also lies in $\mathfrak{D}$.

transformations. For the first assertion observe that $a\in\mathfrak{O}$ means $H(a)\neq\infty$. Thus $H(d^{-1}ad)\neq\infty$ for each $d\in D^*$ by assumption (iii), consequently $d^{-1}ad\in\mathfrak{O}$. In the second case observe $H(u)\neq 0$ for $u\in U$. If $d^{-1}ud\notin U$ then $H(d^{-1}ud)=0$, consequently $H(d^{-1}u^{-1}d)=\infty$, by assumption (ii). Consequently $H(u^{-1})=\infty$, by (iii), that is, $u\notin\mathfrak{O}$ contrary to the assumption on u. Now let $d\in D^*$, then either $H(d)\neq\infty$ or $H(d)=\infty$. By construction of $\mathfrak{O}$ we have $d\in\mathfrak{O}$ if $H(d)\neq\infty$ and, by (ii), $d^{-1}=a\in\mathfrak{O}$ if $H(d)=\infty$. Thus each element of D^* is a quotient of elements in $\mathfrak{O}$. Finally let a, $b\in D^*$, then at least one of the pairs $\{ab^{-1}, b^{-1}a\}$, $\{a^{-1}b, ba^{-1}\}$ lies in $\mathfrak{O}$. Without loss of generality we may assume $H(ab^{-1})\neq\infty$. Then $H(b^{-1}ab^{-1}b)=H(b^{-1}a)\neq\infty$ by assumption (iii). Therefore $\mathfrak{O}$ is a valuation ring by Lemma 5.

Let Δ be an isolated subgroup of Γ. Then a proper restatement of the commutative proof may be used to show the following lemma.

LEMMA 6. *The prime ideals $\mathfrak{p}$ of the valuation ring $\mathfrak{O}$ are in* 1-1 *correspondence with the isolated subgroups Δ of Γ.*

COROLLARY. *The invariant isolated subgroups of Γ are in* 1-1 *correspondence with the prime ideals of $\mathfrak{O}$ which are invariant under all inner automorphisms of D.*

LEMMA 7. *Each invariant prime ideal $\mathfrak{p}$ of $\mathfrak{O}$ determines a quotient ring $\mathfrak{O}_\mathfrak{p}$ in which the extended ideal $\mathfrak{p}\mathfrak{O}_\mathfrak{p}$ is a prime ideal whose residue class ring $\mathfrak{O}_\mathfrak{p}/\mathfrak{p}\mathfrak{O}_\mathfrak{p}$ is a division ring.*

PROOF. Let $\mathfrak{O}_\mathfrak{p}$ be the set $\{ab^{-1}; a\in\mathfrak{O}, b\in\mathfrak{O}-\mathfrak{p}\}$. Then $\mathfrak{O}_\mathfrak{p}$ may alternately be defined as the set of quotients $b^{-1}a$, say $\mathfrak{O}'$. Observe $b^{-1}a=a^{-1}(ab^{-1})a=a(a^{-1}ba)^{-1}$ for $a\neq\mathfrak{O}$ with $a^{-1}ba\notin\mathfrak{O}-\mathfrak{p}$ for $\mathfrak{p}$ is an invariant prime ideal. Thus $\mathfrak{O}'\subseteq\mathfrak{O}_\mathfrak{p}$ and conversely. Next $\mathfrak{O}_\mathfrak{p}$ is an invariant subring of D, for $d^{-1}(ab^{-1})d=(d^{-1}ad)(d^{-1}bd)^{-1}$ where $d^{-1}ad\in\mathfrak{O}$, $d^{-1}bd\in\mathfrak{O}-\mathfrak{p}$ with $d\in D^*$ since $\mathfrak{O}$ and $\mathfrak{p}$ are invariant. It remains to show that $\mathfrak{O}_\mathfrak{p}$ is a ring. Let $a_1b_1^{-1}$, $a_2b_2^{-1}$ be two elements of $\mathfrak{O}_\mathfrak{p}$. Without loss of generality it may be assumed that $V(b_2)\leqq V(b_1)$, then $a_1b_1^{-1}+a_2b_2^{-1}=(a_1+a_2b_2^{-1}b_1)b_1^{-1}$ where $b_2^{-1}b_1\in\mathfrak{O}$. Consequently the sum lies in $\mathfrak{O}_\mathfrak{p}$. For the product observe $a_1b_1^{-1}\cdot a_2b_2^{-1}=a_1b_1^{-1}b_2^{*-1}a_2$ with $b_2^*\in\mathfrak{O}-\mathfrak{p}$ by the invariance of $\mathfrak{O}$ and $\mathfrak{p}$. Next $a_1(b_2^*b_1)^{-1}a_2=b_3^{-1}a_1a_2^*\in\mathfrak{O}_\mathfrak{p}$ with $b_3\in\mathfrak{O}-\mathfrak{p}$, $a_2^*\in\mathfrak{O}$, by the invariantive properties.

Consider next the extended set $\mathfrak{p}\mathfrak{O}_\mathfrak{p}=\{\sum_{j=1}^{N}p_ja_jb_j^{-1}$ with $p_j\in\mathfrak{p}$, $a_jb_j^{-1}\in\mathfrak{O}_\mathfrak{p}\}$. Then $p_ja_jb_j^{-1}=x_jb_j$ with $x_j=p_ja_jb_j^{-1}p_j^{-1}=(p_ja_jp_j^{-1})(p_j^{-1}b_jp_j)^{-1}$ with the first factor in $\mathfrak{O}$ and the second in $\mathfrak{O}-\mathfrak{p}$. Thus $\mathfrak{p}\mathfrak{O}_\mathfrak{p}=\mathfrak{O}_\mathfrak{p}\mathfrak{p}$. The general element $\sum p_ja_jb_j^{-1}$ may be written as $p_1\sum(p_1^{-1}p_j)a_jb_j^{-1}=p_1s_1$, $s_1\in\mathfrak{O}_\mathfrak{p}$, for it may be assumed without loss of generality that

$V(p_1) \leqq V(p_j), j=2, \cdots, N$. As an immediate consequence it follows that $\mathfrak{p}\mathfrak{O}_\mathfrak{p}$ is an invariant set for $d^{-1}(p_1s_1)d = (d^{-1}p_1d)(d^{-1}s_1d) \in \mathfrak{p}\mathfrak{O}_\mathfrak{p}$ since $\mathfrak{p}$ and $\mathfrak{O}_\mathfrak{p}$ were recognized to be invariant sets. Let $p_1s_1, p_2s_2 \in \mathfrak{p}\mathfrak{O}_\mathfrak{p}$. Without loss of generality it may be assumed that $V(p_1) \leqq V(p_2)$. Then $p_1s_1+p_2s_2 = p_1(s_1+p_1^{-1}p_2s_2)$ where $p_1^{-1}p_2 \in \mathfrak{p}$, therefore $s_1+p_1^{-1}p_2s_2 \in \mathfrak{O}_\mathfrak{p}$ and $p_1s_1+p_2s_2 \in \mathfrak{p}\mathfrak{O}_\mathfrak{p}$. Finally $\mathfrak{p}\mathfrak{O}_\mathfrak{p} \cdot \mathfrak{O}_\mathfrak{p} \subseteq \mathfrak{p}\mathfrak{O}_\mathfrak{p}$ and $\mathfrak{O}_\mathfrak{p} \cdot \mathfrak{O}_\mathfrak{p}\mathfrak{p} \subseteq \mathfrak{O}_\mathfrak{p}\mathfrak{p}$, that is, $\mathfrak{p}\mathfrak{O}_\mathfrak{p}$ is a two-sided ideal. To show that $\mathfrak{p}\mathfrak{O}_\mathfrak{p}$ is a prime ideal it suffices to prove that $a_1b_1^{-1}, a_2b_2^{-1} \in \mathfrak{O}_\mathfrak{p} - \mathfrak{p}\mathfrak{O}_\mathfrak{p}$ implies $a_1b_1^{-1} \cdot a_2b_2^{-1} \in \mathfrak{O}_\mathfrak{p} - \mathfrak{p}\mathfrak{O}_\mathfrak{p}$. Observe $a_1, a_2 \in \mathfrak{O} - \mathfrak{p}$, that is, $a_1a_2 \in \mathfrak{O} - \mathfrak{p}$. Therefore $a_1b_1^{-1} \cdot a_2b_2^{-1} = a_1a_2(b_2 \cdot a_2^{-1}b_1a_2)^{-1}$ where $b_2, a_2b_1a_2^{-1} \in \mathfrak{O} - \mathfrak{p}$ for $\mathfrak{p}$ is an invariant prime ideal of $\mathfrak{O}$. Consequently $a_1b_1^{-1} \cdot a_2b_2^{-1} \in \mathfrak{O}_\mathfrak{p} - \mathfrak{p}\mathfrak{O}_\mathfrak{p}$. Let $\mathfrak{O}_\mathfrak{p}/\mathfrak{p}\mathfrak{O}_\mathfrak{p} = D_0$, and suppose $c_0 \in D_0^*$. Then there exist a, b so that $(ab^{-1})_0 = c_0$ with $a \notin \mathfrak{p}$. Hence $(ab^{-1})^{-1}ba^{-1} \notin \mathfrak{p}\mathfrak{O}_\mathfrak{p}$, thus $(ba^{-1})_0 \in D_0$ and $c_0(ba^{-1})_0 = (1)_0$.

Now it is possible to carry over the results of the commutative case.

LEMMA 8. *Each invariant prime ideal* $\mathfrak{p}$ *of* $\mathfrak{O}$ *with the associated isolated subgroup* Δ *of* Γ *determines a valuation* V_Δ *in* $D_\Delta = \mathfrak{O}_\mathfrak{p}/\mathfrak{p}\mathfrak{O}_\mathfrak{p}$ *whose valuation ring is* $\mathfrak{O}/\mathfrak{p}$ *and whose value group is* Δ. *More specifically,* $V_\Delta(a_0) = V(a)$ *where* a mod $\mathfrak{p}\mathfrak{O}_\mathfrak{p} = a_0 \in D_\Delta$.

LEMMA 9. *The quotient ring* $\mathfrak{O}_\mathfrak{p}$ *is a valuation ring of* D *whose value group is* Γ/Δ *letting* $V_{\Gamma/\Delta}(a) = V(a)$ mod Δ, *and whose associated residue algebra is* D_Δ.

The preceding lemmas may be combined so as to give the following theorem.

THEOREM 2. *Relative to each isolated invariant subgroup* Δ *of* Γ *the valuation* V *can be split into* (i) *a valuation* $V_{\Gamma/\Delta}$ *of* D *and* (ii) *a valuation* V_Δ *of the residue algebra* D_Δ *for* $V_{\Gamma/\Delta}$ *with its value group equal to* Δ.

REMARK. The prime ideals $\bar{\mathfrak{p}}_\nu$ of D_Δ in $\mathfrak{O}/\mathfrak{p}$ arise exactly as the homomorphic images of the prime ideals $\mathfrak{p}_\nu \subseteq \mathfrak{O}$ with $\mathfrak{p}_\nu \supset \mathfrak{p}$. Since every invariant subgroup of Δ is not an invariant subgroup of Γ it will in general not be true that an invariant prime $\bar{\mathfrak{p}}_\nu$ is the homomorphic image of an invariant prime ideal $\mathfrak{p}_\nu$ of $\mathfrak{O}$.

THEOREM 3. *Suppose* D *has a valuation* V_1 *with value group* Γ_1 *and residue class algebra* D_1. *Then each valuation* V_2 *in* D_1 *with value group* Γ_2 *whose valuation ring has an invariant image in* D *determines a valuation* V *on* D *with value group* Γ *so that* Γ *contains an order isomorphic image of* Γ_2 *with* $\Gamma/\Gamma_2 \cong \Gamma_1$. *Moreover the residue class ring of* D *with respect to* V *is equal to the residue class ring of* D_1 *with respect to* V_2.

The proof of the theorem involves a direct restatement of the proof for the parallel theorem involving a commutative field.

The preceding discussion of a division algebra with a valuation V may be utilized to establish the existence of a wide variety of algebras with prescribed value groups and algebras of residue classes. Suppose that $\overline{D}$ is a division algebra which is to be the algebra of residue classes for a valuation V with value group Γ. Noting that in a given algebra D the elements $d \in D^*$ induce by $a \pmod{\mathfrak{P}} \rightarrow d^{-1}ad \pmod{\mathfrak{P}}$, $a \in \mathfrak{O}$, automorphisms on the algebra of residues $\overline{D}$, one is led to the following construction. Assume that each $\gamma \in \Gamma^+$ induces an automorphism $\bar{a} \rightarrow \bar{a}^\gamma$ in $\overline{D}$ and let to γ be associated a symbol $t(\gamma)$. Let $\boldsymbol{D}$ be isomorphic to $\overline{D}$. Consider then a group extension of $\boldsymbol{D}^*$ by the group Γ with the defining relations

$$\boldsymbol{a} \rightarrow \boldsymbol{a}^\gamma = t(\gamma)^{-1}\boldsymbol{a}t(\gamma), \qquad t(0) = 1, \qquad t(\gamma)^{-1} = t(-\gamma),$$
$$t(\alpha)t(\beta) = \boldsymbol{f}(\alpha, \beta)t(\alpha + \beta),$$

where the $\boldsymbol{f}(\alpha, \beta)$ satisfy the customary relations for factor sets.[8] Now define D to be the set of all formal power series $A = \sum \boldsymbol{a}_\nu t(\alpha_\nu)$ where $\boldsymbol{a}_\nu \in \boldsymbol{D}$ and $\{\alpha_\nu\}$ is a well ordered montonically increasing sequence in Γ with a finite first element. If $B = \sum \boldsymbol{b}_\nu t(\beta_\nu)$ is another element of D then $A+B$ is to be the series obtained by adding the coefficients at identical marks. The product AB is to be defined by formal multiplication observing that $\boldsymbol{a}t(\alpha)\boldsymbol{b}t(\beta) = \boldsymbol{a}\boldsymbol{b}^\alpha \boldsymbol{f}(\alpha, \beta)t(\alpha+\beta)$. Thus the system D becomes a ring without divisors of zero. Next define $V(A) = \alpha_1 = V(t(\alpha_1))$ where $A = \sum \boldsymbol{a}_\nu t(\alpha_\nu) = [\sum \boldsymbol{a}_\nu t(\alpha_\nu)t(\alpha_1^{-1})]t(\alpha_1)$ $= [\sum \boldsymbol{a}_\nu \boldsymbol{f}(\alpha_\nu, -\alpha_1)t(\alpha_\nu - \alpha_1)]t(\alpha_1)$, $\alpha_\nu - \alpha_1 \geqq 0$. Then $V(\boldsymbol{a}) = 0$ for $\boldsymbol{a} \in \boldsymbol{D}^*$. The function V satisfies the postulates for a valuation. Observe that $A = \boldsymbol{a}t(\alpha) + A_0$, $B = \boldsymbol{b}t(\beta) + B_0$ with $V(A_0) > \alpha$, $V(B_0) > \beta$, respectively, imply $AB = \boldsymbol{a}\boldsymbol{b}^\alpha \boldsymbol{f}(\alpha, \beta)t(\alpha+\beta) + C_0$ where $V(C_0) > \alpha+\beta$. Hence $V(AB) = \alpha+\beta$. Finally $V(A+B) \geqq \min [V(A), V(B)]$. For the proof one may assume without loss of generality that $\beta \geqq \alpha$. Then $A+B$ $= [\boldsymbol{a} + A_0 t(\alpha)^{-1} + \boldsymbol{b}\boldsymbol{f}(\beta, -\alpha)t(\beta-\alpha) + B_0 t(\alpha)^{-1}]t(\alpha)$ which proves the triangle inequality. It remains to show that each element $A \in D^*$ has an inverse. Observe that each sequence $\{\sum_{j=0}^{n} C^j,\ n \rightarrow \infty,\ V(C) > 0\}$ has a limit $\sum_{j=0}^{\infty} C^j$ in the set D. Thus $(1+C)^{-1} = \sum_{j=0}(-1)^j C^j$ lies in D. Write $A = \boldsymbol{a}(1+C)t(\alpha)$, $\boldsymbol{a} \in \boldsymbol{D}^*$, then $A^{-1} = t(-\alpha)\sum_{j=0}^{\infty}(-1)^j C^j \boldsymbol{a}^{-1}$. It is shown directly that the valuation ring $\mathfrak{O} = \{\sum a_\nu t(\alpha_\nu),\ \alpha_\nu \geqq 0\}$

[8] See, for example, H. Zassenhaus, *Lehrbuch der Gruppentheorie*, Leipzig, 1937; A. A. Albert, *Structure of algebras*, Amer. Math. Soc. Colloquium Publications, vol. 24, 1939. Factor sets were first used in the valuation theory by Kaplansky, loc. cit. pp. 315–317.

contains the prime ideal $\mathfrak{P}=\{\sum a_\nu t(\alpha_\nu),\ \alpha_\nu>0\}$ so that $\mathfrak{O}/\mathfrak{P}\cong \boldsymbol{D}\cong\overline{D}$. Moreover $\mathfrak{O}$ and $\mathfrak{P}$ are invariant subsets of D.

DEFINITION 2. *A division algebra D is termed relatively complete with respect to a valuation V if each of its subfields K containing the center Z is relatively complete with respect to the induced valuation V_K.*

LEMMA 10. *A relatively complete division algebra of finite rank n^2 over its center Z has a commutative value group with respect to the given valuation.*

PROOF. Observe first that the valuation V induces a non-trivial valuation V_Z on Z. For let $d^m+z_1d^{m-1}+\cdots+z_m=0$, $z_j\in Z$, be the irreducible equation satisfied by an element $d\in D$ with $V(d)>0$. Then $V(z_m)=V(d)+V(d^{m-1}+z_1d^{m-2}+\cdots+z_{m-1})$. Thus $V(z_m)=V_Z(z_m)>0$ in case $V_Z(z_j)\geqq 0$, $j=1,\cdots,m-1$, by the triangle inequality for valuations. In case some $V_Z(z_j)<0$ nothing is to be proved. Since Z is, by hypothesis, relatively complete with respect to V_Z the usual theory of prolongation for valuations may be applied.[9] Set $V^*(a)=n^{-1}V_Z(Na)$ where N denotes the reduced norm of D/Z. Then $V^*(a)=V(a)$ for otherwise the subfields $Z(a)$, Z would be relatively complete with respect to two inequivalent valuations. Hence Z would be algebraically complete contrary to the hypothesis that D is a division algebra.[10] Next let $a, b\in D$, then $V(ab)=V^*(ab)=V^*(ba)=V(ba)$, that is, the value group Γ of V is abelian.

DEFINITION 3. *A relatively complete division algebra D is termed algebraic if the algebra $D(a, b)$ generated by any two elements $a, b\in D$ over Z has finite rank over its center $Z(a, b)$.*

THEOREM 4. *The value group of an algebraic relatively complete division algebra D is abelian.*

PROOF. Let α, β be two elements of Γ. Suppose that a, b are any two elements of D with $V(a)=\alpha$, $V(b)=\beta$. By hypothesis the algebra $D(a, b)$ has finite rank over the relatively complete field $Z(a, b)$. Hence, by Lemma 10, $V(ab)=V(ba)$, that is, $\alpha+\beta=\beta+\alpha$.

REMARK. The preceding theorem indicates that a division algebra with a noncommutative value group must contain elements which are transcendental over its center.

ILLUSTRATIVE EXAMPLE. Let Γ be the lexicographically ordered group of motions in the plane, that is, the group of all couples of real

[9] See, for example, O. F. G. Schilling, loc. cit. pp. 568–570 and Schilling, *Normal extensions of relatively complete fields*, Amer. J. Math. vol. 65 (1943) pp. 309–334.

[10] See Albert, loc. cit.

numbers (α, β) subject to the following law of combination: if (γ, δ) is a second pair then $(\alpha, \beta)+(\gamma, \delta)=(\alpha+\gamma, e^{\gamma}\beta+\delta)$. The set Γ^+ of positive elements consists of all couples for which either $\alpha>0$ or $\alpha=0$ and $\beta>0$. Observe the following facts: (i) $-(\alpha,\beta)=(-\alpha, -\beta e^{-\alpha})$, (ii) the set $\{(0, \beta)\}$ is an invariant isolated subgroup Δ, (iii) $\Gamma/\Delta \cong\{\alpha\}$, and (iv) Γ has no proper center. Let F be a field and t a transcendental element over F. Consider the set D of all formal power series $\sum_{(\alpha,\beta)}a_{\alpha,\beta}t^{(\alpha,\beta)}$ where $a_{\alpha,\beta}\in F$ and the elements (α, β) form well ordered monotonically increasing sequences. Set $t^{(0,0)}=1$ and $t^{(\alpha,\beta)}t^{(\gamma,\delta)}=t^{\xi}$, $\xi=(\alpha, \beta)+(\gamma, \delta)$. Then D is a division algebra with Γ as a value group. The valuation ring $\mathfrak{O}$ consists of all series $\sum a_{\alpha,\beta}t^{(\alpha,\beta)}$ where all $(\alpha, \beta)\geqq(0, 0)$; the prime ideal $\mathfrak{P}$ contains all series with $(\alpha, \beta)>(0, 0)$. Both $\mathfrak{O}$ and $\mathfrak{P}$ are invariant subsets for the group of inner automorphisms of D^*, and $\mathfrak{O}/\mathfrak{P}\cong F$. Corresponding to Δ there is an invariant prime ideal $\mathfrak{p}$ in $\mathfrak{O}$ whose associated valuation ring maps homomorphically on the field of all formal series $\overline{F}_\Delta = \{\sum_\beta a_\beta \bar{t}^{(0,\beta)}\} = \{\sum a_\beta \bar{t}_1^\beta\}$. Select now in D the subfield $F_\Delta=\{\sum a_\beta t^{(0,\beta)}\} = \{\sum a_\beta t_1^\beta\}$. Next set $t^{(\alpha,0)}=t_2^\alpha$, then $t^{(\alpha,\beta)}=t_2^\alpha t_1^\beta$. The rule of combination in Γ implies $t_1^\beta t_2^\alpha=t_2^\alpha t_1^\mu$ with $\mu=e^\alpha\beta$. Since each element of D may now be expressed as $\sum_{-\infty<\alpha_\nu}A_{\alpha_\nu}t_2^{\alpha_\nu}$, $A_{\alpha_\nu}\in F_\Delta$, $\{\alpha_\nu\}$ increasing, it can be seen that D is a crossed extension of F_Δ by the set $\{t_2^\alpha\}$. The associated factor set is equal to unity for $t_2^\alpha t_2^\gamma=t_2^{\alpha+\gamma}=t_2^{\gamma+\alpha}=t_2^\gamma t_2^\alpha$. This interpretation of the algebra D as a transcendental crossed extension of the algebra F_Δ by means of an extension of the value group of F_Δ can be generalized in several directions. As in the construction of an algebra for a given algebra of residue classes $\overline{D}$ and a value group Γ one may introduce factor sets. Moreover statements can be made for an algebra D whose value group Γ possesses a chain of normal isolated subgroups Δ_j whose factor groups Γ_j/Γ_{j+1} are isomorphic to abelian ordered groups. To obtain explicit results it is useful to assume that (i) the successive algebras of residue classes D_{j+1} have isomorphic images $\phi_j(D_{j+1})$ in D_j, (ii) the isomorphism ϕ_j commutes with the group of inner automorphisms of D_j, and (iii) D_j is maximally complete with respect to the valuation having Γ_j/Γ_{j+1} for its value group.[11]

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[11] Observe in this connection the results of Kaplansky, loc. cit.