THE EXISTENCE OF ANORMAL CHAINS

DAVID BLACKWELL

1. Introduction. Let $\mathcal{B}$ be a Borel field of subsets of a space $X$, and let $P(x, E)$ be for fixed $x$ a probability measure on $\mathcal{B}$ and for fixed $E$ a $\mathcal{B}$-measurable function of $x$. $P(x, E)$ may be considered as representing the transition probability of going from $x$ into $E$ in a single trial. Denote by $\Omega$ the space of sequences $\omega: (x_0, x_1, \cdots)$ where $x_i \in X$ and by $\mathcal{E}$ the Borel field of subsets of $\Omega$ determined by all sets

$$\{ x_i \in E \}, \quad \text{where} \quad E \in \mathcal{B}, \quad i = 1, 2, \cdots .$$

Doob [2, pp. 102–103]¹ has shown that there exists for each $x \in X$ a probability measure $P_x(S)$ defined on $\mathcal{E}$ such that for every $P_x$-integrable function $f(x_1, \cdots , x_n)$

$$\int f(\omega)dP_x = \int \cdots \int f(x_1, \cdots , x_n)dP(x_{n-1}, x_n) \cdots dP(x, x_1),$$

that $\Omega$ with the measure $P_x$ is a Markoff process, that is, $E(x_1, \cdots , x_n; g) = E(x_n; g)$ where $g = g(x_{n+1}, x_{n+2}, \cdots )$ and the $E$'s denote conditional expectations with respect to the indicated variables, and that $E(x_1, \cdots , x_r; f)$ is the function obtained by carrying out the first $n-r$ integrations in (1).

Write $Q(x, E) = P_x(\lim \sup \{ x_i \in E \})$, so that $Q(x, E)$ represents the probability of entering $E$ infinitely often, starting from $x$. Following Doblin [1, p. 68 et seq.] we make the following definitions for sets of $\mathcal{B}$: $E$ is inessential if $Q(x, E) = 0$ for all $x$, and essential otherwise. An essential set is improperly essential if it is a denumerable sum of inessential sets, and absolutely essential otherwise. A finite or denumerable sum of improperly essential sets is consequently improperly essential. $E$ is closed if $P(x, E) = 1$ for all $x \in E$, and a closed set is indecomposable if it does not contain two disjunct non-empty closed subsets. An absolutely essential indecomposable set is said to be normal if it contains a closed set which contains no improperly essential subsets and anormal otherwise. If $X$ is a normal set, we shall say that the Markoff chain determined by $P(x, E)$ is a normal chain.

Doblin [1] has obtained for normal chains many elegant results which are considerably more complicated for the anormal case. For example [1, p. 81] in the normal case there exists a closed set $G$ such

¹ Numbers in brackets refer to the references cited at the end of the paper.

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that $Q(x, E) = 1$ for every essential subset $E$ of $G$ and every $x \in G$; in the anormal case $G$ can no longer be chosen independently of $E$. It is consequently of interest to investigate conditions for normality. Doblin [1, p. 82] has given a quite general sufficient condition for the occurrence of the normal case, and no example of an anormal chain has hitherto been given. The purpose of this paper is to give a simple necessary and sufficient condition for the occurrence of the normal case, consisting merely in the measurability of the function $f(x)$ which represents the probability that, starting from $x$, the point remains indefinitely in an improperly essential set. An example of an anormal chain is also given.

2. The normal case. We restate a result of Doblin [1, p. 80] in the following theorem.

**Theorem 1.** If $X$ is indecomposable and absolutely essential, the closed set $\{Q(x, E) = 1\}$ is non-empty if and only if $E$ is absolutely essential.

The following lemma asserts that the probability of entering $E$ infinitely often, starting from $y$ at the $j$th trial, is independent of $j$:

**Lemma.** With respect to any $P_x$ measure,

$$P_x(x_j = y; \limsup \{x_i \in E\}) = Q(y, E).$$

**Proof.** If $f_{m,n}$ is the characteristic function of the set $E(m, n) = \sum_{m \leq i \leq n} \{x_i \in E\}$, it follows from (1) that

$$P_y(E(m, n)) = \int \int \cdots \int f_{m,n}dP(x_{n-1}, x_n) \cdots dP(y, x_1)$$

$$= P_x(x_j = y; E(m + j, n + j)).$$

Letting first $n$, then $m$, become infinite, we obtain (2).

The following theorem, which is new and of some independent interest, asserts that for any two sets $T$ and $E$, unless there are points of $T$ which are practically certain to enter $E$ infinitely often, it is impossible for a point to enter both $T$ and $E$ infinitely often.

**Theorem 2.** If $Q(x, E) \leq a < 1$ for all $x \in T$, then for all $x$

$$h(x) = P_x(\limsup \{x_i \in E\} \cdot \limsup \{x_i \in T\}) = 0.$$

**Proof.** Define $E(N, n) = \{x_i \in E \text{ for } N \leq i < n, x_n \in E\}$, $T(N, n) = \{x_i \in T \text{ for } N \leq i < n, x_n \in T\}$. Now for fixed $N$,

$$h(x) \leq \sum_{N < n < r} P_x(E(N, n)T(n, r)) \limsup \{x_i \in E\}.$$
But
\[ P_x(E(N, n)T(n, r) \limsup \{ x_i \in E \}) \]
\[ = \int_{E(N, n)T(n, r)} P_x(x_1, \ldots, x_r) \limsup \{ x_i \in E \} dP_x \]
\[ = \int_{E(N, n)T(n, r)} Q(x_r, E) dP_x \leq aP_x(E(N, n)T(n, r)), \]
by the lemma and the fact that \( x_r \) belongs to \( T \) in the domain of integration. Using the last inequality in (4), we obtain
\[ h(x) \leq a \sum_{N<n<r} P_x(E(N, n)T(n, r)) = aP_x \left( \sum_{N<n<r} E(N, n)T(n, r) \right). \]
Letting \( N \) become infinite we obtain \( h(x) \leq ah(x) \), which implies (3). Our necessary and sufficient condition for the occurrence of the normal case, given in Theorem 3, is an easy consequence of Theorems 1 and 2.

**Theorem 3.** Let \( X \) be absolutely essential and indecomposable. Then \( X \) is normal if and only if \( f(x) = \text{l.u.b.} \text{ imp. ess.} E \) \( Q(x, E) \) is \( \mathcal{B} \)-measurable.

**Proof.** By Theorem 1, for every improperly essential \( E \) and every \( x \) we have \( Q(x, E) < 1 \). It follows that \( f(x) < 1 \) for all \( x \), since a denumerable sum of improperly essential sets is improperly essential. If \( f(x) \) is measurable, there exists an \( a < 1 \) such that \( T = \{ f(x) < a \} \) is absolutely essential; for \( X = \sum_{n=1}^{\infty} \{ f(x) < 1 - 1/n \} \), and not all these sets can be improperly essential. Denoting by \( S \) the closed set \( \{ Q(x, T) = 1 \} \), it follows from Theorem 1 that \( S \) is non-empty and from Theorem 2 that \( Q(x, E) = 0 \) for all \( x \in S \) and all improperly essential \( E \). In particular if \( E \subseteq S \) and \( E \) is not absolutely essential, \( Q(x, E) = 0 \) for all \( x \in E \), which by Theorem 2 in the special case \( T = E \) implies that \( E \) is inessential. Thus \( S \) is a non-empty closed set containing no improperly essential subsets, and \( X \) is normal.

Conversely if there exists such a subset \( S \) it is easily verified that \( f(x) = Q(x, X - S) \) and is therefore measurable.

**3. An anormal chain.** The space \( X \) is the semi-infinite interval \( 0 \leq x < \infty \), and \( \mathcal{B} \) is the Borel field of all finite or denumerable subsets of \( X \) and their complements. Let \( a_n \) be any sequence of numbers such that \( 0 < a_n < 1, \prod_0^\infty a_n > 0 \). We define
\[ P(x, E) = a_n f(x + 1, E) + (1 - a_n) d(E), \]
where \( n \) is the largest positive integer not exceeding \( x \), \( d(E) \) is 1 if \( E \) is non-denumerable and 0 otherwise, and \( f(x, E) \) is the characteristic function of \( E \). \( P(x, E) \) is clearly a probability measure on \( \mathcal{B} \) for fixed \( x \).

To verify that \( P(x, E) \) is \( \mathcal{B} \)-measurable for fixed \( E \), we may assume that \( E \) is at most denumerable, since \( P(x, CE) = 1 - P(x, E) \). For this case \( P(x, E) = 0 \) except on an at most denumerable set and is consequently measurable. Since the probability of going from \( x \) into \( x+1 \) is \( a_n \) for \( n \leq x < n+1 \) and since \( \prod_0^n a_n > 0 \), every set containing all points \( x+n, n = 1, 2, \ldots \), for some \( x \) is essential. The closed sets are those nondenumerable sets which contain with \( x \) all points \( x+n \), so that \( X \) is indecomposable. Finally \( X \) is absolutely essential, since \( \sum S_n = X \) implies that some \( S_n \) is nondenumerable and hence essential. Thus \( X \) is anormal.

References


Howard University