BOOK REVIEWS

Theory of games and economic behavior. By John von Neumann and Oskar Morgenstern. Princeton University Press, 1944. 18+625 pp. $10.00.

Posterity may regard this book as one of the major scientific achievements of the first half of the twentieth century. This will undoubtedly be the case if the authors have succeeded in establishing a new exact science—the science of economics. The foundation which they have laid is extremely promising. Since both mathematicians and economists will be needed for the further development of the theory it is in order to comment on the background necessary for reading the book. The mathematics required beyond algebra and analytic geometry is developed in the book. On the other hand the non-mathematically trained reader will be called upon to exercise a high degree of patience if he is to comprehend the theory. The mathematically trained reader will find the reasoning stimulating and challenging. As to economics, a limited background is sufficient.

The authors observe that the give-and-take of business has many of the aspects of a game and they make an extensive study of the strategy of games with this similarity in mind (hence the title of this book). In the game of life the stakes are not necessarily monetary; they may be merely *utilities*. In discussing utilities the authors find it advisable to replace the questionable marginal utility theory by a new theory which is more suitable to their analysis. They note that in the game of life as well as in social games the players are frequently called upon to choose between alternatives to which probabilities rather than certainties are attached. The authors show that *if a player can always arrange such fortuitous alternatives in the order of his preferences, then it is possible to assign to each alternative a number or numerical utility expressing the degree of the player's preference for that alternative. The assignment is not unique but two such assignments must be related by a linear transformation.*

The concept of a game is formalized by a set of postulates. Even the status of information of each player on each move is accounted for and is characterized by a partition of a certain set. *The amount which player k receives at the conclusion of the play is a function* $\mathfrak{F}_k(\sigma_1, \sigma_2, \cdots, \sigma_\nu)$ *of the moves* $\sigma_1, \sigma_2, \cdots, \sigma_\nu$ *where some of the* σ's *may be the moves of chance* (dealing cards, throwing dice, and so on).

The concept of a game admits of a rather drastic simplification which practically relieves the players of the necessity of playing.

Imagine that all possible strategies of all players have been catalogued. Then player k can tell his secretary that he wishes to play strategy τ_k. When she looks up this strategy she finds a complete prescription determining every move for every possible eventuality. Thus the secretaries could get together and determine the outcome of the game if they could only find an equitable method of accounting for the moves of chance. But chance enters into the game very much as one of the players. Thus we can imagine a cataloguing of the possible strategies of chance. Suppose for the moment that the strategy τ_0 of chance has been decided upon and that the players have chosen respectively the strategies $\tau_1, \tau_2, \cdots, \tau_n$. Then the strategies determine the moves. Hence $\mathfrak{F}_k(\sigma_1, \sigma_2, \cdots, \sigma_\nu)$ *is a function* $\mathfrak{G}_k(\tau_0, \tau_1, \cdots, \tau_n)$ *of the strategies* and the outcome of the game is determined. But how should τ_0 be selected? Instead of selecting τ_0 the secretaries could *assign to each player k the amount* $\mathfrak{H}_k(\tau_1, \tau_2, \cdots, \tau_n)$ *which he would receive on the average if strategies* $\tau_1, \tau_2, \cdots, \tau_n$ *of the players were chosen*. The amount $\mathfrak{H}_k$ is the mathematical expectation of $\mathfrak{G}_k$. It is computed in terms of the probabilities of the various strategies τ_0 and these probabilities are in turn computed in terms of the probabilities of the moves of chance.

The game has now been reduced to one in which each player makes just one move—the selection of a strategy. Each player makes his move in complete ignorance of the moves of the other players. The authors have accomplished this simplification of the game with complete rigor and with complete adherence to the rules laid down by the postulates.

A 1-player game corresponds to the economy of a man on a desert island. It is the Robinson Crusoe economy or a strictly regimented communism. If the player is wise, he will choose his strategy τ_1 so that $\mathfrak{H}_1(\tau_1)$ is a maximum. This is the only case where a game is settled by simple maximum considerations.

An n-player zero-sum game is one for which the sum of the $\mathfrak{F}_k$'s is zero for all choices of $\sigma_1, \sigma_2, \cdots, \sigma_\nu$ and hence one for which the sum of the $\mathfrak{H}_k$'s is zero for all choices of $\tau_1, \tau_2, \cdots, \tau_n$. Social games are zero-sum but the game of economics is decidedly not zero-sum since society as a whole can improve its status if all members behave properly. *However an arbitrary n-player game can be reduced to a zero-sum* $(n+1)$*-player game* by introducing a fictitious player $n+1$ who receives the amount $\mathfrak{H}_{n+1}(\tau_1, \tau_2, \cdots, \tau_n)$ which is the negative of the total received by the remaining n players. Note that the functions $\mathfrak{H}_k$ do not contain the variable τ_{n+1}, that is, the fictitious player is not permitted to choose a strategy. It will appear later that further restrictions on the activities of this player are necessary to prevent his influencing the outcome of the game.

A zero-sum 2-player game Γ can be characterized by a single function $\mathfrak{H}(\tau_1, \tau_2) = \mathfrak{H}_1(\tau_1, \tau_2)$ since $\mathfrak{H}_2(\tau_1, \tau_2) = -\mathfrak{H}(\tau_1, \tau_2)$ by virtue of the relation $\mathfrak{H}_1 + \mathfrak{H}_2 = 0$. In this game player 1 will attempt to maximize $\mathfrak{H}$ (or $\mathfrak{H}_1$) whereas player 2 will attempt to minimize $\mathfrak{H}$ (or maximize $\mathfrak{H}_2$). Since these are diametrically opposed tendencies, it looks as though nothing could be decided. However we can gain insight into the problem by considering a modified game Γ_1 which is the same as Γ except that player 1 moves first and player 2 knows 1's move. In Γ_1 after player 1 chooses τ_1, player 2 will choose τ_2 so as to minimize $\mathfrak{H}$. It is therefore advisable for 1 to choose τ_1 so as to maximize $\min_{\tau_2} \mathfrak{H}(\tau_1, \tau_2)$ where $\min_{\tau_2} \mathfrak{H}(\tau_1, \tau_2)$ is the minimum with respect to τ_2 of $\mathfrak{H}(\tau_1, \tau_2)$. Player 1 will then receive.

$$v_1 = \max_{\tau_1} \min_{\tau_2} \mathfrak{H}(\tau_1, \tau_2)$$

and 2 will receive $-v_1$. Next consider a third game Γ_2 which is the same as Γ except that 2 moves first and 1 knows 2's move. If both players of Γ_2 are skillful, then 1 will receive the amount

$$v_2 = \min_{\tau_2} \max_{\tau_1} \mathfrak{H}(\tau_1, \tau_2)$$

and 2 will receive $-v_2$. In the original game Γ if both players are skillful, 1 will receive at least v_1 and at most v_2 whereas 2 will receive at least $-v_2$ and at most $-v_1$. *Hence* $v_1 \leqq v_2$ *and these quantities are bounds for the outcome of the game.* If $v_1 = v_2$, the game is determined but in general this is not the case.

Note that Γ reduces to Γ_1 if 2 discovers 1's strategy whereas Γ becomes Γ_2 if 1 discovers 2's strategy. Hence it is advisable for the players to conceal their strategies. The concealment is accomplished by using probabilities. Thus 1 chooses τ_1 with probability ξ_{τ_1} and 2 chooses τ_2 with probability η_{τ_2}. *The average outcome* $K(\xi, \eta)$ *for player 1 is the mathematical expectation of* $\mathfrak{H}(\tau_1, \tau_2)$ *with respect to the probabilities* ξ_{τ_1} *and* η_{τ_2} where ξ is the vector with components $\xi_1, \xi_2, \cdots$ and η is the vector with components $\eta_1, \eta_2, \cdots$. The introduction of these probabilities modifies Γ and consequently modifies Γ_1, Γ_2 and the bounds v_1, v_2. The new bounds become

$$v_1' = \min_{\xi} \max_{\eta} K(\xi, \eta)$$

and

$$v_2' = \min_{\eta} \max_{\xi} K(\xi, \eta).$$

It is easily shown that $v_1 \leqq v_1' \leqq v_2' \leqq v_2$, that is, that each player is at least as well off as before the probabilities were introduced. Moreover it can be shown that

$$v_1' = v_2' = v$$

and hence that the game is determined. The proof of the latter result depends on the fact that the numbers $x_{\tau_2} = \sum_{\tau_1} \mathfrak{H}(\tau_1, \tau_2)\xi_{\tau_1}$ are components of a vector χ which depends on ξ and that the tips of the vectors χ for all possible ξ's constitute a convex set of points.

Next consider an n-player game in which the players divide into two hostile groups called S and $-S$. This can be interpreted as a 2-player game between the players S and $-S$. If probabilities are employed in the manner described above, then S will receive

$$v(S) = v_1' = v_2' = v$$

and $-S$ will receive

$$v(-S) = -v(S).$$

If I is the set of all players, then $v(I) = 0$, that is, the game is zero-sum. Finally

$$v(S + T) \geqq v(S) + v(T)$$

if S and T are mutually exclusive groups. That is, the players of $S+T$ can obtain at least as much by cooperating as they can by splitting up into two groups S and T. The function $v(S)$ satisfying the above relations is called a characteristic function. Corresponding to any function satisfying these relations there exists a game having this $v(S)$ as its characteristic function. The construction of such a game involves partitions of I into subsets called rings and solo sets.

If the equality $v(S+T) = v(S) + v(T)$ always holds, that is, if $v(S)$ is additive, then the coalitions will be ineffective and the game will be determined. This is the case for $n = 2$. Moreover two characteristic functions (whether additive or not) which differ by an additive function will produce the same strategies of coalitions. If $v(S)$ is not additive, it can be modified by a suitable additive function and a suitable scale factor so that $v(S) = -1$ for all 1-element sets. Thus for $n = 3$, $v(S)$ is given by the following table

$$v(S) = \begin{cases} 0 \\ -1 \\ +1 \\ 0 \end{cases} \textit{for the} \begin{cases} 0\textit{-element set } (-I \textit{ or the complement of } I) \\ 1\textit{-element sets} \\ 2\textit{-element sets (complements of } 1\textit{-element sets)} \\ 3\textit{-element set } (I). \end{cases}$$

For $n \geqq 4$, $v(S)$ is no longer determined and the number of possibilities becomes almost bewildering. The reader will begin to realize that there is never a dull moment with these games. We have seen that for each of the cases $n=1, 2, 3, 4$ a new situation appears. For $n=5$ no new phenomenon has as yet been discovered but for $n \geqq 6$ we first meet the possibility of a game which splits into two or more games which are in some respects quite distinct but which nevertheless exert potent influences on one another. This phenomenon has the counterpart of nations whose economies are distinct yet interdependent.

It remains to consider what coalitions can be expected to form in a given game and how the stakes will be divided in the presence of such coalitions. A division of stakes is called an *imputation* and is represented by a vector α with components $\alpha_1, \alpha_2, \cdots, \alpha_n$ where α_k is the amount the kth player receives. One could imagine that if a group of novices were playing one of these games a certain chaos would result. Coalitions would be made and broken as each player sought to improve his own status. Finally as the players became more acquainted with the game certain imputations would come to be trusted because of the stability of the corresponding coalitions and because of the profitableness to an effective group of players. There would thus emerge *a set V of trusted imputations*. There would of course be players who were dissatisfied with any given trusted imputation but they would not be strong enough to force a change unless they could bribe some of the favored players to desert their coalitions. Nor would such bribery be effective since the potential recipient of the bribe would realize that the chaos produced by his desertion would eventually leave him in a less favorable position. Thus V corresponds to a group behavior pattern. It is an institution or a morality arising from enlightened self interest.

But how can V be described mathematically? We begin with a definition. *We say that an imputation α dominates an imputation β if there is an effective group of players each of which is better off under α than under β. The group is effective provided it can guarantee for its members the stakes prescribed by α against any opposition from without the group.* A set V of imputations is called a solution provided every imputation outside of V is dominated by some imputation of V and no imputation in V is dominated by any other imputation in V. Thus *V is a maximal set of mutually undominated imputations.* Unfortunately dominance does not produce even a partial ordering of the set of all imputations. It is not a transitive relation. This makes the discovery of solutions a difficult task. We shall however outline a method of finding solutions for the case $n=3$.

When $n=3$ we have

$$\alpha - (\alpha_1, \alpha_2, \alpha_3) \quad \text{with} \quad \alpha_1 + \alpha_2 + \alpha_3 = 0,$$

that is, the game is zero-sum. Thus the tip of α lies in a plane which passes through the origin and is equally inclined to the coordinate axes. This plane is divided into six congruent sectors by the traces of the coordinate planes. Next $\alpha_k \geqq -1$ (for $k=1, 2, 3$) since each player can obtain at least -1 without the benefit of any coalition (see the above table). These inequalities require the tip of α to lie within an equilateral triangle whose center is at the common intersection of the traces of the coordinate planes and whose sides are parallel to these traces. An imputation α dominates those imputations which are represented by points interior to three parallelograms each of which has two sides in common with the above equilateral triangle and one vertex at the tip of α. On the basis of these geometrical considerations it is easy to find solutions V. We first look for a V whose imputations do not all lie on a line α_k = a constant (that is, a line parallel to a trace). There is only one such solution, namely,

$$V\colon \quad (1/2, 1/2, 0), \qquad (1/2, 0, 1/2), \qquad (0, 1/2, 1/2).$$

We next look for a V whose imputations do lie on a line, say, $\alpha_3 = c$. The corresponding solutions are

$$V_c\colon \quad (a, -a-c, c)$$

where a and c are required to satisfy certain inequalities. Thus V_c contains a continuum of solutions corresponding to values of the parameter a. This exhausts the possible solutions. The first solution V seems quite reasonable whereas V_c seems unnatural and difficult to interpret but let us return to this question later.

Let us consider the following non-zero-sum 2-player game. Each player (1 or 2) chooses either the number 1 or the number 2. If both players choose 1, then each receives the stake 1/2. Otherwise each receives -1. If we reduce this game to a zero-sum 3-player game by the introduction of a fictitious player 3, then the characteristic function becomes the one given in the above table. Now if we take the first solution V, we discover that the fictitious player may play an active part in the formation of coalitions. Hence if we wish to retain the 2-player character of the game, we must choose the solution V_c and it is reasonable to assign to c the value -1.

The authors apply this theory of games to the analysis of a market consisting of one buyer and one seller and also of a market consisting of two buyers and one seller.

The book leaves much to be done but this fact only enhances its interest. It should be productive of many extensions along the lines of economic interpretation as well as of mathematical research. In fact the authors suggest a number of directions in which research might profitably be pursued.

ARTHUR H. COPELAND

Principles of stellar dynamics. By S. Chandrasekhar. The University of Chicago Press, 1942. 10+251 pp.

The primary field of this book is astronomy and not mathematics, although the latter is used as an essential tool. The readers of this review, professional mathematicians almost exclusively, will have a normal human interest in the major astronomical aspects of the book, but their critical scrutiny is bound to be concentrated on how the astronomical problems are formulated mathematically and what sort of mathematics has been proposed for their solution. For this reason, and partly also in the interest of brevity, this review treats only of the mathematical aspects of the book.

In the first chapter is given a detailed discussion of the kinematical concepts appropriate to the study of stellar systems. Since these systems contain a large number of stars, it becomes necessary to introduce a method similar to that employed in hydrodynamics, where the motion of a fluid is described by a vector field, representing at each point and for each instant of time the velocity of the fluid. In hydrodynamics the velocity of the fluid at a point is conceived as the velocity of the "fluid particle" at the point in question. But this notion of a "particle" at the point in question is difficult to make precise, especially if one assumes the fluid to consist of a large number of small atoms with relatively large empty spaces between them. Nevertheless such a concept (in which the stars play the role of the atoms) is characteristic of stellar dynamics as distinguished from celestial (particle) mechanics, which considers systems containing but a relatively small number of bodies.

The components $U_0(x, y, z, t)$, $V_0(x, y, z, t)$, $W_0(x, y, z, t)$ of the vector field thus introduced do not, of course, necessarily represent the components of velocity of a star which might happen to be at the point (x, y, z), but rather the velocity of the centroid of stars in a "small volume" about the point (x, y, z). The components of velocity of an individual star are written in the form $U=U_0+u$, $V=V_0+v$, $W=W_0+w$, where the vector (u, v, w) is called the *residual* velocity. The statistical consideration of these residual velocities is a characteristic of stellar dynamics and gas theory as distinguished from