This gives an application of the methods recently discovered by the author to lattice points in the region $|x^n+y^n|\leqq 1$ where $m=p/q>4$ and p, q are odd integers. (Received April 9, 1945.)

ANALYSIS

109. R. P. Agnew: *Tauberian theorems for Nörlund summability.*

It is shown that a Tauberian theorem for Nörlund summability given by R. P. Cesco (Universidad Nacional de La Plata, Publicaciones de la Facultad de Ciencias Fisicomatematicas (2) vol. 4 (1944) pp. 443–445) and more general theorems are implied by familiar Tauberian theorems for Abel summability. (Received May 14, 1945.)

110. R. H. Cameron: *Some examples of Fourier-Wiener transforms of analytic functionals.*

Let $F[x]=F[x(\cdot)]$ be a functional which is defined throughout the space K of complex continuous functions $x(t)$ defined on $0\leqq t\leqq 1$ and vanishing at $t=0$, and let F possess the property that $F[x+iy]$ is Wiener summable in x over C for each fixed $y(\cdot)$ in K. (C is the subspace of all real functions in K.) Then the functional $G[y]=\int_C^W F[x+iy]d_Wx$ is called the Fourier-Wiener transform of $F(x)$. Examples are given of various functionals $F(x)$ which have the property that the transforms of their transforms exist and equal $F(-x)$. The fact that there exist extensive classes of functionals having this property is proved in a paper by W. T. Martin and the author. (Received May 9, 1945.)

111. R. H. Cameron and W. T. Martin: *Fourier-Wiener transforms of analytic functionals.*

Let K be the space of all continuous complex functions $x(t)$ defined on $0\leqq t\leqq 1$ which vanish at $t=0$, let C be the subset of all real functions of K, and let $F[x]$ be a functional defined on K. Then, as defined by one of the authors in another paper, $G(y)=\int_C^W F(x+iy)d_Wx$ is called the Fourier-Wiener transform of $F(x)$ if it exists for all y in K. The purpose of the present paper is to exhibit three extensive classes of functionals which are taken into themselves in a one-to-one manner by the Fourier-Wiener transformation in such a way that the inverse transformation is $F(x)=\int_C^W G(y-ix)d_Wy$. In particular, this property is enjoyed by the class E_m of functionals $F(x)$ which are mean continuous, of mean exponential type, and "entire," that is, continuous in the topology defined by root mean square distance between functions, bounded by A exp $[B\int_0^1 x(t)^2dt]^{1/2}$, and "entire" in the sense that $F(x+\lambda y)$ is entire in λ for all x and y in K. (Received May 9, 1945.)

112. Nelson Dunford and Robert Schatten: *On the associate and conjugate space for the direct product of Banach spaces.*

The direct product $E_1\otimes_N E_2$ of two Banach spaces E_1, E_2 (Trans. Amer. Math. Soc. vol. 53 (1943) pp. 195–217), which naturally depends on the norm N, determines uniquely an "associate space" $\bar{E}_1\otimes_{\bar{N}}\bar{E}_2$ ($\bar{N}$ is the norm "associate" with N) and a conjugate space $\overline{E_1\otimes_N E_2}$. It is shown that for a "natural crossnorm," $\bar{L}\otimes_{\bar{N}}\bar{L}$ is a proper subset of $\overline{L\otimes_N L}$. Similarly, for a "natural crossnorm," $\bar{l}\otimes_{\bar{N}}\bar{l}$ is a proper subset of $\overline{l\otimes_N l}$. A related example of a non-reflexive crossspace $E_1\otimes_N E_2$ is constructed, for which E_1, E_2, and N are all reflexive. (Received April 23, 1945.)

113. Herbert Federer: *Coincidence functions and their integrals.*

Two functions f and g are said to coincide at (x, y) if and only if $f(x)=g(y)$. If f parametrizes a k-dimensional surface in Euclidean n-space, and g an $(n-k)$-dimensional surface, the number of coincidences of f and g is the number of intersections of these surfaces. Now keep the first surface fixed and move the second rigidly; or, otherwise said, superimpose upon g an arbitrary isometric transformation S of n-space. Count the number of coincidences of f with the superposition $(S:g)$, that is, the number of intersections of the fixed and the movable surface. Integrate this count over the group of all isometric transformations of n-space with respect to its Haar measure, properly and explicitly normalized. If f and g are sufficiently regular, say Lipschitzian, the value of the integral is the product of the areas of the two surfaces times a number $\beta(n, k)$, which depends only on n and k. This result is established, and $\beta(n, k)$ is evaluated. Further it is proved that the area of a sufficiently regular k-dimensional surface in n-space equals $\beta(n, k)^{-1}$ times the average area of its projection into a k-dimensional subspace of n-space. This, in turn, suggests the definition of a lower semicontinuous area for all continuous k-dimensional surfaces in terms of the stable values of their projections into k-dimensional subspaces. (Received April 11, 1945.)

114. Evelyn Frank: *Corresponding type continued fractions.*

This paper is concerned with a development of properties of *corresponding type continued fractions* ("C-fractions"), $1+a_1z^{\alpha_1}/1+a_2z^{\alpha_2}/1+\cdots$, first studied by Leighton and Scott (Bull. Amer. Math. Soc. vol. 45 (1939) pp. 596–605) and Scott and Wall (Ann. of Math. vol. 41 (1940) pp. 328–349). (i) There is an algorithm analogous to that given by Wall for J-fractions (Bull. Amer. Math. Soc. vol. 51 (1945) pp. 97–105) for expanding an arbitrary power series $P(z)=1+c_1z+c_2z^2+\cdots$ into a C-fraction. (ii) If (1) $\alpha_1\geqq\omega+1, \sum_0^p\alpha_{2k+1}-1\geqq\sum_1^p\alpha_{2k}+\omega\geqq\sum_1^p\alpha_{2k-1}, p=1, 2, 3, \cdots;$ $\omega\geqq 0$, then the approximants of the C-fraction are all Padé approximants for $P(z)$. Incidentally, an extension is made of a known theorem on the Padé table. (iii) Necessary and sufficient conditions on the c_p for the α_p to satisfy (1) are given, which reduce to well known conditions in case $\alpha_p=1$. (iv) Certain transformations of Stieltjes (*Œuvres*, vol. 2) are extended to C-fractions for which (1) holds. (Received April 12, 1945.)

115. Fritz Herzog and J. D. Hill: *The Bernstein polynomials for discontinuous functions.*

The well known theorem of Weierstrass on the uniform approximation of continuous functions by polynomials was proved by S. Bernstein as follows: If $f(x)$ is continuous in $[0, 1]$ then the polynomials $B_n(f; x)=\sum_{k=0}^n f(k/n)C_{n,k}x^k(1-x)^{n-k}$ converge to $f(x)$ uniformly in $[0, 1]$. These polynomials, called the *Bernstein polynomials* of f, are determined for a function $f(r)$, defined only for rational numbers r in $[0, 1]$. Such an $f(r)$ will be called a *skeleton*. This paper proposes to investigate the Bernstein polynomials for arbitrary skeletons. For bounded skeletons upper and lower bounds for $\liminf_n B_n(f; x)$ and $\limsup_n B_n(f; x)$ are obtained for each x. Let $\mathfrak{S}$ be the class of skeletons $f(r)$ for which $\lim_n f(r_n)$ exists whenever $r_n\to x-$ or $r_n\to x+$, x being any point in $[0, 1]$. For skeletons of the class $\mathfrak{S}$, $\{B_n(f; x)\}$ converges in $[0, 1]$, and this convergence is uniform in any closed subinterval in which the limit function is continuous. The problem of when two skeletons in $\mathfrak{S}$ lead to the same limit function is also completely answered. Convergence conditions of a more special character are given as well as examples which illustrate the theory developed in the paper. (Received April 25, 1945.)

116. J. D. Hill: *Summability of sequences of* 0'*s and* 1'*s.*

Let T denote a regular matrix method of summability and $\mathfrak{X}$ the class of all sequences $x\equiv\{\alpha_k\}$, $\alpha_k=0$ or 1, with infinitely many $\alpha_k=1$. A result of Steinhaus states that corresponding to each T there is at least one x in $\mathfrak{X}$ which is not summable T. Let $\mathfrak{X}_0(T)$ be the set of all such x and $\mathfrak{X}_1(T)$ its complement with respect to $\mathfrak{X}$. Let $\mathfrak{X}_2(T)$ be the subset of $\mathfrak{X}_1(T)$ composed of all x which are summable T to 1/2. A one-to-one mapping of $\mathfrak{X}$ into the interval $\mathfrak{Y}\equiv(0<y\leqq 1)$ is obtained by defining y as the dyadic fraction $0.\alpha_0\alpha_1\alpha_2\cdots$ if $x=(\alpha_0,\alpha_1,\alpha_2,\cdots)$ is a point of $\mathfrak{X}$. The map of $\mathfrak{X}_i(T)$ is denoted by $\mathfrak{Y}_i(T)$ for $i=0, 1, 2$. A result of Borel is equivalent to the statement that the Lebesgue measure of $\mathfrak{Y}_2(C_1)$ is one, where C_1 is the Cesàro method of order one. The author investigates further questions suggested by the results of Steinhaus and Borel. The principal results obtained are: (1) For each T the sets $\mathfrak{Y}_1(T)$ and $\mathfrak{Y}_2(T)$ have the same measure which is either zero or one. (2) For each T the set $\mathfrak{Y}_1(T)$ is of the first category and hence $\mathfrak{Y}_0(T)$ is of the second category. (3) A sufficient condition is given that $\mathfrak{Y}_2(T)$ have measure one. (4) By (3) it is shown that $\mathfrak{Y}_2(T)$ has measure one if T is either Cesàro, Euler, or Borel summability. (Received May 15, 1945.)

117. A. P. Hillman: *Complex zeros of the Bessel-Weber functions* Y_n.

It is shown that for integral n the Bessel-Weber functions Y_n (for the definition see G. N. Watson, *A treatise on the theory of Bessel functions*, p. 64) have no negative real zeros and that each branch of Y_n has complex zeros whose real parts tend to minus infinity. Numerical values of the first few zeros of Y_0, Y_1, and Y_1' as well as asymptotic expansions for the zeros of Y_n and Y_n' are also given. Nothing seems to have been known previously about zeros of Y_n in the left half-plane, not even for Y_0 or Y_1. (Received May 17, 1945.)

118. C. N. Moore: *Convergence factors in general analysis.* II.

In a paper presented at the summer meeting of 1944 (cf. Bull. Amer. Math. Soc. Abstract 50-9-221) the author developed certain theorems in the field of general analysis which include as particular cases some of the theorems concerning convergence factors in infinite series which may be found in vol. 22 of the American Mathematical Society's Colloquium Publications and analogous theorems concerning infinite integrals. In the present paper the theory is further developed so as to include additional special results in the field of infinite series and infinite integrals. (Received May 14, 1945.)

119. George Polya and Gabor Szegö: *On finite Fourier integrals.*

This paper deals with three independent problems connected by their relation to finite Fourier integrals. 1. As an analogue of a theorem of Fejér-F. Riesz on non-negative trigonometric polynomials the following is proved. Let $f(u)$ be L-integrable in $-a$, a, $F(x)=\int_{-a}^{a}f(u)e^{iux}du$. Let $F(x)$ be L-integrable in $-\infty$, ∞, and $F(x)\geqq 0$ for all real x. Then an L-integrable function $\phi(u)$ exists such that, putting $\Phi(x)=\int_0^a\phi(u)e^{iux}du$, we have $F(x)=|\Phi(x)|^2$. Various applications are given, for instance the inequality $F(x)\leqq af(0)$. 2. This part deals with the characterization of entire functions defined by a finite Hankel integral $x^{-\nu}\int_0^a f(u)J_\nu(xu)du$. The results are generalizations of those due to Paley-Wiener and Plancherel-Polya corresponding to the cases $\nu=\pm 1/2$. Applications are given to the evaluation of various integrals involving Bessel functions. 3. As an analogue of a well known minimum property of the partial sums of a Fourier series, the following is proved. Let $(f(u))^2$ be L-integrable in $-\infty$, ∞, $F(x)\sim\int_{-\infty}^{\infty}f(u)e^{iux}du$. Let $g(u)$ be an arbitrary L-integrable function in $-a$, a,

$G(x)=\int_{-a}^{a}g(u)e^{iux}du$. For all functions $g(u)$ the integral $\int_{-\infty}^{\infty}(F(x)-G(x))^2dx$ is a minimum if $g(u)=f(u)$ (except on a zero set). (Received April 12, 1945.)

120. V. C. Poor: *Complex functions possessing differentials.*

The purpose of this paper is to study the differentials of complex functions, in particular the restricted Hamilton differential which was shown elsewhere to be equivalent to the Rainich differential and also the Young differential and their relations to each other. The Young definition is also given by Mrs. Young in discussing functions possessing differentials. In her paper necessary and sufficient conditions for the existence of a Young differential of a complex function are given. However, some of these conditions are superfluous, and the theorem, properly formulated, is not proved. The necessary and sufficient conditions for the existence of a Young differential will be set up here and proved. Also certain results of the Fundamenta paper will be generalized to the complex plane. Finally, a power series expansion of a polygenic function will be given. (Received April 2, 1945.)

121. V. C. Poor: *On the Hamilton differential.*

This note concerns a modified form of the Hamilton differential, due to G. Y. Rainich, which is the Hamilton differential modified by a linearity condition. Necessary and sufficient conditions for the existence of this differential of a function on a vector n-space are set up and proved. It is shown that the modified definition possesses the linearity property. (Received April 2, 1945.)

122. Raphael Salem and Antoni Zygmund: *Capacity of sets and Fourier series.*

Beurling showed that, if $\sum n(a_n^2+b_n^2)$ is finite, the set of the points of divergence of the trigonometric series (*) $a_0/2+\sum(a_n \cos nx+b_n \sin nx)$ is of logarithmic capacity zero. He also stated without proof that, if $\sum n^{\alpha-\epsilon}(a_n^2+b_n^2)$ converges for some $0<\alpha<1$ and for every $\epsilon>0$, the capacitary dimension of the set of points where the harmonic function $\sum(a_n \cos nx+b_n \sin nx)r^n$ has no finite radial limit is greater than or equal to $1-\alpha$. The latter result may be generalized as follows. If $\sum n^{\alpha}(a_n^2+b_n^2)<\infty$, the set of the points of divergence of (*) is of $(1-\alpha)$-capacity zero. (Received May 8, 1945.)

123. Raphael Salem and Antoni Zygmund: *The approximation by partial sums of Fourier series.*

Let $f(x)$ be a function of period 2π and of the class Lip α, $0<\alpha<1$. It is familiar that the partial sums $s_n(x)$ of the Fourier series of f satisfy the condition $s_n-f=O(n^{-\alpha}\log n)$, and that the logarithm here cannot be removed. (a) The estimate cannot be improved even if in addition to the previous hypotheses it is assumed that f is of bounded variation. (b) If, however, not only the function f but also its absolute variation is of the class Lip α, $0<\alpha<1$, then $s_n(x)-f(x)=O(n^{-\alpha})$. (Received May 8, 1945.)

124. I. M. Sheffer: *Note on Appell polynomials.*

Thorne has recently given (Amer. Math. Monthly vol. 52 (1945) pp. 191–193) an interesting characterization of Appell polynomials by means of a Stieltjes integral. In this note is given a companion representation of Appell sets in terms of a Stieltjes integral. The result is extended to the case of polynomial sets of type zero. For the Appell set $\{P_n(x)\}$ the representation is $P_n(x)=\int_0^{\infty}\{(x+t)^n/n!\}d\beta(t)$, where $\beta(t)$ is a function of bounded variation on $(0, \infty)$, for which the moment constants $b_n = \int_0^\infty t^n d\beta(t)$ exist, with $b_0 \neq 0$. (Received May 10, 1945.)

125. Otto Szász: *On the absolute convergence of trigonometric series.*

Generalizing a theorem of Fatou on trigonometric series with monotonically decreasing coefficients, the author proves the following theorem: If $\rho_n > 0$, $\rho_{n+1} \leqq c\rho_n$, c a constant, and if the trigonometric series $\sum \rho_n \cos nx$ is absolutely convergent at one point x_0, then $\sum \rho_n < \infty$. The same is true for the sine series if in addition $x_0 \not\equiv 0 \pmod{\pi}$. The proof is quite short and elementary. The author extends this result to series of the type $\sum \rho_n \cos \lambda_n x$, $\sum \rho_n \sin \lambda_n x$, where $0 < \lambda_1 < \lambda_2 < \cdots$. (Received May 10, 1945.)

126. W. J. Trjitzinsky: *Integral equations in problems of representation of functions of a complex variable.*

This work relates to the representation of functions of a complex variable, more general than analytic, in terms of "Cauchy double integrals." (Received May 16, 1945.)

Applied Mathematics

127. H. E. Salzer: *Formulas for direct and inverse interpolation of a complex function tabulated along equidistant circular arcs.*

When an analytic function $f(z)$ may be approximated by a complex polynomial of degree $n-1$ passing through the values of the function at n points, according to the Lagrange-Hermite interpolation formula, it often happens that those n points are situated along the arc of a circle (equally spaced) and it is required to obtain $f(z)$ for z off the circle but near the arguments. An important case is when $f(z)$ is tabulated in polar form (including tabulation along the vertices of any regular polygon). The formulas that were obtained will facilitate direct interpolation when $f(z)$ is known at three, four, or five points. They furnish the real and imaginary parts of $L_k^{(n)}(P)$, where $f(z) \sim \sum L_k^{(n)}(P) f(z_k)$, as functions of $P^m = p_m + iq_m$ and θ. Here $P = (z - z_0)/h$, h being the distance between successive points z_k, and θ denotes the angle between successive chords joining the points z_k. For extensive use for a fixed θ, one can readily obtain $L_k^{(n)}(P)$ in the form $\sum C_k^{(n)}(p_m + iq_m)$. A method for inverse interpolation is given, employing the coefficients of the polynomials $L_k^{(n)}(P)$ in an expansion derived in the author's *A new formula for inverse interpolation*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 513–516. (Received May 18, 1945.)

128. H. E. Salzer: *Table of coefficients for double quadrature without differences, for integrating second order differential equations.*

On the basis of a double quadrature of the Lagrange interpolation formula, a table of coefficients has been computed to determine a function at equally spaced points (to within an arbitrary $Ax+B$), when its second derivative is known at those points. The coefficients cover the cases where the second derivative may be approximated by a polynomial ranging from the second to tenth degrees (that is, from three-point through eleven-point formulas), and are given exactly. Their chief value will occur in the numerical solution of ordinary linear differential equations of the second order, which can always be reduced to the form $y'' + g(x)y = h(x)$. They can also be employed to integrate the more general equation $y'' + \phi(x, y) = 0$. In every case it is necessary to begin with a few values of y'' which can always be found by the usual methods.