THE BASIS THEOREM FOR VECTOR SPACES OVER RINGS

C. J. EVERETT

It is the purpose of this note to establish the following theorem:

THEOREM. *A vector space* $M = u_1K + \cdots + u_mK$ *of* m *basis elements over a ring* $K = \{0, a, b, \cdots, 1\}$ *with unit* 1 *has the property that every subspace* $N > 0$ *possesses a basis of* $n \leqq m$ *elements if and only if* K *is a right principal-ideal-ring without zero-divisors.*

That such a ring insures the basis condition for subspaces is well known [3, p. 121].[1]

Suppose now that every subspace $N > 0$ has a basis of $n \leqq m$ elements. It has been shown [2, Theorem (F)] that every right ideal $R > 0$ of K must then have a single generator: $R = r_0K$, where $r_0k = 0$ implies $k = 0$. Moreover, since every right ideal has a finite set of generators, the ascending chain condition must hold for right ideals of K [3, p. 26]. It therefore suffices to prove the following two lemmas.

LEMMA 1. *In a ring* K *with unit* 1 *and ascending chain condition for right ideals, equations* $ab = 1$, $ac = 0$ *imply* $c = 0$.

If $c \neq 0$, the linear transformation $k \rightarrow ak$, $k \in K$, would be of type (iv) [2, p. 313], that is, $K/K_0 \cong K$, and $0 < K_0 < K_1 < K_2 < \cdots$, where K_i is defined inductively as the set of all elements of K mapped into elements of K_{i-1}. This contradicts the chain condition.

LEMMA 2. *A ring* K *with unit in which every right ideal* $R > 0$ *is of the form* r_0K, *where* $r_0k = 0$ *implies* $k = 0$, *has no zero divisors.*

Let $sc = 0$, $s \neq 0$, and $sK = r_0K \neq 0$, where $r_0k = 0$ implies $k = 0$. We have $s = r_0a$, $r_0 = sb = r_0 \cdot ab$, $r_0(ab - 1) = 0$, and hence $ab = 1$. Also, $sc = 0 = r_0ac$, and $ac = 0$. Since Lemma 1 applies to K, $c = 0$.

It should be noted that the result follows also from a result of Baer's [1, Theorem 5 or Lemma 4] which states that in a ring with unit and weak maximal condition, $ab = 1$ implies $ba = 1$.

BIBLIOGRAPHY

1. R. Baer, *Inverses and zero-divisors*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 630–638.

Presented to the Society, November 25, 1944; received by the editors February 14, 1945.

[1] Numbers in brackets refer to the bibliography.

2. C. J. Everett, *Vector spaces over rings*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 312–316.

3. B. L. van der Waerden, *Moderne Algebra*, vol. 2, 1st ed., Berlin, 1931.

UNIVERSITY OF WISCONSIN

ON A CONSTRUCTION FOR DIVISION ALGEBRAS OF ORDER 16

R. D. SCHAFER

It is not known whether there exist division algebras of order 16 (or greater) over the real number field $\mathfrak{R}$. In discussing the implications of this question in algebra and topology, A. A. Albert told the author that the well known Cayley-Dickson process[1] does not yield a division algebra of order 16 over $\mathfrak{R}$ and suggested a modification of that process which might. It is the purpose of this note to show that, while Albert's construction can in no instance yield such an algebra over $\mathfrak{R}$, it does yield division algebras of order 16 over other fields, in particular the rational number field R.

Initially consider an arbitrary field F. Let C be a Cayley-Dickson division algebra of order 8 over F. Define[2] an algebra of order 16 over F with elements $c=a+vb$, $z=x+vy$ (a, b, x, y in C) and with multiplication given by

$$cz = (a + vb)(x + vy) = (ax + g\cdot ybS) + v(aS\cdot y + xb) \tag{1}$$

where S is the involution $x\rightleftarrows xS=t(x)-x$ of C and g is some fixed element of C. The Cayley-Dickson process is of course the instance $g=\gamma$ in F.

For A to be a division algebra over F the right multiplication[1] R_z must be nonsingular for all $z\neq 0$ in A. Now

$$R_z = \begin{pmatrix} R_x & SR_y \\ SL_yL_g & L_x \end{pmatrix}$$

Received by the editors January 19, 1945, and, in revised form, March 19, 1945.

[1] See [1] and [2] for background and notations. Numbers in brackets refer to the references cited at the end of the paper.

[2] We should remark that this modification of the Cayley-Dickson process does yield non-alternative division algebras of orders 4 and 8 over $\mathfrak{R}$ when applied to the algebras of complex numbers and real quaternions instead of to C. See R. H. Bruck, *Some results in the theory of linear non-associative algebras*, Trans. Amer. Math. Soc. vol. 56 (1944) pp. 141–199, Theorem 16C, Corollary 1, for a generalization.