A REMARK ON A RESULT DUE TO BLICHFELDT

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Let $\sigma \geqq 1$ and $\xi_1, \cdots, \xi_n$ be $n \geqq 3$ linear forms of the real variables $x_1, \cdots, x_n$ of nonvanishing determinant Δ. For simplicity's sake we assume $|\Delta| = 1$. Let $2s$ of the forms be pairwise conjugate complex and the remaining $n-2s$ be real. Then

$$| \xi_1 |^\sigma + \cdots + | \xi_n |^\sigma \leqq 1$$

defines a symmetric convex body in the x-space, the volume $V(\sigma)$ of which equals

$$2^n \cdot \frac{\{\Gamma(1 + \alpha)\}^{n-2s}\{\pi\Gamma(1 + 2\alpha)/2^{1+2\alpha}\}^s}{\Gamma(1 + n\alpha)} \qquad (\alpha = 1/\sigma).$$

Minkowski's principle states that there is a lattice point $(x_1, \cdots, x_n) \neq (0, \cdots, 0)$ satisfying the inequality

$$| \xi_1 |^\sigma + \cdots + | \xi_n |^\sigma \leqq r^\sigma$$

provided

$$r^n \geqq 2^n V^{-1}(\sigma). \tag{1}$$

By means of Blichfeldt's method, van der Corput and Schaake[1] obtained a sharpening of this result for $\sigma \geqq 2$. Decisive in this procedure is an inequality of the following form

$$\sum_{p,q=1}^{k} | z_p - z_q |^\sigma \leqq \epsilon(\sigma) k \cdot \sum_{p=1}^{k} | z_p |^\sigma, \tag{2}$$

where the factor $\epsilon(\sigma)$ depends neither on the arbitrary complex numbers z_p nor on k. Once such an inequality is known, (1) may be replaced by

$$r^n \geqq (\epsilon(\sigma))^{n/\sigma} \cdot \frac{n + \sigma}{\sigma} \cdot V^{-1}(\sigma). \tag{3}$$

The elementary relation

$$| u - v |^\sigma \leqq 2^{\sigma-1}(| u |^\sigma + | v |^\sigma)$$

(following from the fact that x^σ is a convex function of $x > 0$) implies (2) with $\epsilon(\sigma) = 2^\sigma$. Substituted in (3) this does not improve, but on

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[1] Acta Arithmetica vol. 2 (1936) pp. 152–160.

the contrary worsens, Minkowski's inequality. However, van der Corput and Schaake obtained the better value $2^{\sigma-1}$ for $\sigma \geqq 2$. I shall show here that $\epsilon(\sigma)=2$ is a legitimate choice for $1 \leqq \sigma \leqq 2$ and that both facts follow almost immediately from Marcel Riesz's convexity theorem.

Indeed, specialize this theorem (Theorem 296 on p. 219 of Hardy, Littlewood and Pólya's *Inequalities*) by taking $\gamma=\alpha$ and the X as the linear forms $X_{pq}=z_p-z_q$. It then turns out that the logarithm of the maximum $M_k(\alpha)$ of

$$\left\{ \sum_{p,q=1}^{k} \left| z_p - z_q \right|^{1/\alpha} \Big/ k \sum_{p=1}^{k} \left| z_p \right|^{1/\alpha} \right\}^{\alpha}$$

for fixed k and variable $z_1, \cdots, z_k$ is a convex function of α in the interval $0 \leqq \alpha \leqq 1$. One readily verifies that

$$M_k(0) = 2, \qquad M_k(1/2) = 2^{1/2}, \qquad M_k(1) = 2(1 - 1/k) \leqq 2.$$

As

$$\left\{ \sum_{p} \left| z_p \right|^{1/\alpha} \right\}^{\alpha} \to \max \left| z_p \right| \quad \text{for} \quad \alpha \to 0,$$

the first equation follows from $\max |z_p - z_q| \leqq 2 \cdot \max |z_p|$ together with the observation that the upper bound 2 is attained for $z_1=1$, $z_2=-1$, $z_3= \cdots =z_k=0$. Similarly the two other equations are immediate consequences of the elementary inequalities

$$\sum_{p,q} \left| z_p - z_q \right|^2 = 2k \sum_{p} \left| z_p \right|^2 - 2 \left| \sum_{p} z_p \right|^2 \leqq 2k \sum_{p} \left| z_p \right|^2,$$

$$\sum_{p \neq q} \left| z_p - z_q \right| \leqq \sum_{p \neq q} \left(\left| z_p \right| + \left| z_q \right| \right) = 2(k-1) \sum_{p} \left| z_p \right|,$$

and the corresponding obvious observations about the z_p for which the upper bound is reached.

Let us use 2 as the basis of our logarithms. Then the values of $\log_2 M_k(\alpha)$ are 1, 1/2 and less than or equal to 1 for $\alpha=0$, 1/2, 1 respectively, and hence the broken line consisting of $1-\alpha$ for $0 \leqq \alpha \leqq 1/2$ and α for $1/2 \leqq \alpha \leqq 1$ gives an upper bound for the convex function $\log_2 M_k(\alpha)$. We thus obtain the promised result that (2) holds with

$$\epsilon(\sigma) = 2^{\sigma-1} \quad \text{for} \quad \sigma \geqq 2 \qquad \text{and} \qquad \epsilon(\sigma) = 2 \quad \text{for} \quad 1 \leqq \sigma \leqq 2. \tag{4}$$

Both choices are the best possible of their kinds, as, for $0 \leqq \alpha \leqq 1/2$, is shown by the example $k=2$, $z_1=-z_2=1$, and, for $1/2 \leqq \alpha \leqq 1$, by the example $z_1=-z_2=1$, $z_3= \cdots =z_k=0$, with large k.

Consider the case $1 \leqq \sigma \leqq 2$. If we substitute the value $\epsilon(\sigma) = 2$ in (3), we shall find that it does not always improve Blichfeldt's known inequality, in particular not for the most interesting case $\sigma = 1$. We observe that

$$\left(\frac{|\xi_1|^\sigma + \cdots + |\xi_n|^\sigma}{n}\right)^{1/\sigma}$$

is an increasing function of the exponent σ, while the upper bound for its lattice minimum as derived from (3), namely,

$$\left(\frac{2}{n}\right)^{1/\sigma} \left(\frac{n+\sigma}{\sigma}\right)^{1/n} (V(\sigma))^{-1/n}, \tag{5}$$

is not. For $s = 0$ the expression (5) tends to a limit with $n \to \infty$, namely

$$\frac{1}{2}\left(\frac{2}{\sigma e}\right)^{1/\sigma} \Big/ \Gamma\left(1 + \frac{1}{\sigma}\right) = 2^{\alpha-1}\left(\frac{\alpha}{e}\right)^{\alpha} \Big/ \Gamma(1+\alpha).$$

The logarithmic derivative of this function with respect to α is negative for $\alpha = 1/2$ and positive for $\alpha = 2/3$, and hence this function has a minimum between $\sigma = 2$ and $\sigma = 1.5$; numerical computation gives as its location $\sigma = \sigma_0 = 1.8653 \cdots$.[2] At this point the value of the function is

$$\leqq 1/(3.146e)^{1/2}$$

which is slightly better than the constant

$$1/(\pi e)^{1/2}$$

due to Blichfeldt.[3]

In conclusion, for $2 \geqq \sigma \geqq \sigma_0$, (1) may be replaced by

$$r^n \geqq 2^{n/\sigma}\left(\frac{n+\sigma}{\sigma}\right) V^{-1}(\sigma),$$

and, for $1 \leqq \sigma \leqq \sigma_0$, (1) may be replaced by

$$r^n \geqq 2^{n/\sigma_0}\left(\frac{n+\sigma_0}{\sigma_0}\right) V^{-1}(\sigma_0).$$

This would be true however σ_0 were chosen within the limits $1 \leqq \sigma_0 \leqq 2$; our special choice approaches the best possible for $n \to \infty$ (and $s = 0$) and is sharp enough to beat Blichfeldt's record by a slight margin, even for small n.

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[2] The author is indebted to Mr. Sze for this numerical value.

[3] Trans. Amer. Math. Soc. vol. 15 (1914) pp. 227–235.