A REMARK ON A RESULT DUE TO BLICHFELDT

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Let \( \sigma \geq 1 \) and \( \xi_1, \ldots, \xi_n \) be \( n \geq 3 \) linear forms of the real variables \( x_1, \ldots, x_n \) of nonvanishing determinant \( \Delta \). For simplicity's sake we assume \( |\Delta| = 1 \). Let \( 2s \) of the forms be pairwise conjugate complex and the remaining \( n - 2s \) be real. Then

\[ |\xi_1|^\sigma + \cdots + |\xi_n|^\sigma \leq 1 \]

defines a symmetric convex body in the \( x \)-space, the volume \( V(\sigma) \) of which equals

\[ 2^n \cdot \frac{\Gamma(1 + \alpha)}{\Gamma(1 + n\alpha)} \frac{\pi^{n-2s} \cdot \pi^{\Gamma(1 + 2\alpha)/2^{1+2\alpha}}}{2^{n-2s}} \alpha = 1/\sigma. \]

Minkowski's principle states that there is a lattice point \( (x_1, \ldots, x_n) \) \( \neq (0, \ldots, 0) \) satisfying the inequality

\[ |\xi_1|^\sigma + \cdots + |\xi_n|^\sigma \leq r^\sigma \]

provided

(1)

\[ r^n \geq 2^n V^{-1}(\sigma). \]

By means of Blichfeldt's method, van der Corput and Schaake obtained a sharpening of this result for \( \sigma \geq 2 \). Decisive in this procedure is an inequality of the following form

(2) \[ \sum_{p, q=1}^{k} |z_p - z_q|^{\sigma} \leq \epsilon(\sigma) k \cdot \sum_{p=1}^{k} |z_p|^{\sigma}, \]

where the factor \( \epsilon(\sigma) \) depends neither on the arbitrary complex numbers \( z_p \) nor on \( k \). Once such an inequality is known, (1) may be replaced by

(3) \[ r^n \geq (\epsilon(\sigma))^{n/\sigma} \cdot \frac{n + \sigma}{\sigma} \cdot V^{-1}(\sigma). \]

The elementary relation

\[ |u - v|^{\sigma} \leq 2^{\sigma-1} (|u|^\sigma + |v|^\sigma) \]

(following from the fact that \( x^\sigma \) is a convex function of \( x > 0 \)) implies (2) with \( \epsilon(\sigma) = 2^\sigma \). Substituted in (3) this does not improve, but on
the contrary worsens, Minkowski's inequality. However, van der Corput and Schaake obtained the better value $2^{e-1}$ for $\sigma \geq 2$. I shall show here that $\epsilon(\sigma) = 2$ is a legitimate choice for $1 \leq \sigma \leq 2$ and that both facts follow almost immediately from Marcel Riesz's convexity theorem.

Indeed, specialize this theorem (Theorem 296 on p. 219 of Hardy, Littlewood and Pólya's Inequalities) by taking $\gamma = \alpha$ and the $X$ as the linear forms $X_{pq} = z_p - z_q$. It then turns out that the logarithm of the maximum $M_k(\alpha)$ of

$$\left\{ \frac{1}{k} \sum_{p=1}^{k} |z_p - z_q|^{1/\alpha} \right\}^{\frac{1}{\alpha}}$$

for fixed $k$ and variable $z_1, \ldots, z_k$ is a convex function of $\alpha$ in the interval $0 \leq \alpha \leq 1$. One readily verifies that

$$M_k(0) = 2, \quad M_k(1/2) = 2^{1/2}, \quad M_k(1) = 2(1 - 1/k) \leq 2.$$  

As

$$\max \left\{ \left( \sum_{p} |z_p|^{1/\alpha} \right)^{\alpha} \right\} \rightarrow \max |z_p| \quad \text{for} \quad \alpha \to 0,$$

the first equation follows from $\max |z_p - z_q| \leq 2 \cdot \max |z_p|$ together with the observation that the upper bound 2 is attained for $z_1 = 1, z_2 = -1, z_3 = \ldots = z_k = 0$. Similarly the two other equations are immediate consequences of the elementary inequalities

$$\sum_{p,q} |z_p - z_q|^2 = 2k \sum_p |z_p|^2 - 2 \left( \sum_p z_p \right)^2 \leq 2k \sum_p |z_p|^2,$$

$$\sum_{p,q} |z_p - z_q| \leq \sum_{p,q} (|z_p| + |z_q|) = 2(k - 1) \sum_p |z_p|,$$

and the corresponding obvious observations about the $z_p$ for which the upper bound is reached.

Let us use 2 as the basis of our logarithms. Then the values of $\log_2 M_k(\alpha)$ are 1, 1/2 and less than or equal to 1 for $\alpha = 0, 1/2, 1$ respectively, and hence the broken line consisting of $1 - \alpha$ for $0 \leq \alpha \leq 1/2$ and $\alpha$ for $1/2 \leq \alpha \leq 1$ gives an upper bound for the convex function $\log_2 M_k(\alpha)$. We thus obtain the promised result that (2) holds with

$$\epsilon(\sigma) = 2^{e-1} \quad \text{for} \quad \sigma \geq 2 \quad \text{and} \quad \epsilon(\sigma) = 2 \quad \text{for} \quad 1 \leq \sigma \leq 2.$$  

Both choices are the best possible of their kinds, as, for $0 \leq \alpha \leq 1/2$, is shown by the example $k = 2, z_1 = -z_2 = 1$, and, for $1/2 \leq \alpha \leq 1$, by the example $z_1 = -z_2 = 1, z_3 = \cdots = z_k = 0$, with large $k$.  

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Consider the case $1 \leq \sigma \leq 2$. If we substitute the value $\varepsilon(\sigma) = 2$ in (3), we shall find that it does not always improve Blichfeldt's known inequality, in particular not for the most interesting case $\sigma = 1$. We observe that

$$
\left( \frac{\xi_1^{\sigma} + \cdots + \xi_n^{\sigma}}{n} \right)^{1/\sigma}
$$

is an increasing function of the exponent $\sigma$, while the upper bound for its lattice minimum as derived from (3), namely,

$$(\frac{2}{n})^{1/\sigma} \left( \frac{n + \sigma}{\sigma} \right)^{1/n} (V(\sigma))^{-1/n},$$

is not. For $s = 0$ the expression (5) tends to a limit with $n \to \infty$, namely

$$\frac{1}{2} \left( \frac{2}{\varepsilon_0} \right)^{1/\sigma} \left( 1 + \frac{1}{\sigma} \right) = 2^{\sigma-1} \left( \frac{\alpha}{\varepsilon} \right)^{\alpha} / \Gamma(1 + \alpha).$$

The logarithmic derivative of this function with respect to $\alpha$ is negative for $\alpha = 1/2$ and positive for $\alpha = 2/3$, and hence this function has a minimum between $\sigma = 2$ and $\sigma = 1.5$; numerical computation gives as its location $\sigma = \sigma_0 = 1.8653 \cdots$. At this point the value of the function is

$$\leq 1/(3.146e)^{1/2}$$

which is slightly better than the constant

$$1/(\pi e)^{1/2}$$

due to Blichfeldt.\(^8\)

In conclusion, for $2 \geq \sigma \geq \sigma_0$, (1) may be replaced by

$$r^n \geq 2^{n/\sigma} \left( \frac{n + \sigma}{\sigma} \right) V^{-1}(\sigma),$$

and, for $1 \leq \sigma \leq \sigma_0$, (1) may be replaced by

$$r^n \geq 2^{n/\sigma_0} \left( \frac{n + \sigma_0}{\sigma_0} \right) V^{-1}(\sigma_0).$$

This would be true however $\sigma_0$ were chosen within the limits $1 \leq \sigma_0 \leq 2$; our special choice approaches the best possible for $n \to \infty$ (and $s = 0$) and is sharp enough to beat Blichfeldt's record by a slight margin, even for small $n$.

\(^8\) The author is indebted to Mr. Sze for this numerical value.