151. A. N. Milgram: *Cyclotomically saturated polynomials and tri-operational algebra.*

Let $f(x)$ be a polynomial whose coefficients are integers mod p where p is a prime number. Call $f(x)$ cyclotomically saturated if it has the property that for each irreducible polynomial $\phi(x)$, if $[\phi(x)]^{\nu} | f(x)$, then also $(x^{p^n}-x)^{\nu} | f(x)$ where n is the degree of $\phi(x)$. In tri-operational algebra (cf. Reports of a Mathematical Colloquium, nos. 5–6, p. 5) Menger raised the question: What polynomials with coefficients over the integers mod p have the property $f(x) | f(g(x))$ for each polynomial $g(x)$? The answer is: $f(x) | f(g(x))$ for each $g(x)$ if and only if $f(x)$ is cyclotomically saturated. (Received June 23, 1945.)

152. J. M. H. Olmsted: *Transfinite rationals.*

As suggested by the treatment of ratios by Eudoxus, two cardinal number pairs, (a, b) and (c, d), are defined to be equivalent if and only if for every pair of cardinal numbers, m and n, ma and nb have the same order relation as mc and nd. Addition, multiplication, division, and ordering are defined among the equivalence classes of cardinal number pairs, the resulting system being a lattice with familiar algebraic laws (for example, multiplication is distributive over addition, joins, and meets). This system is an extension of both the positive rational numbers and the cardinal numbers. Furthermore, it is the smallest extension subject to certain conditions. (Received June 4, 1945.)

153. Gordon Pall: *Hermitian quadratic forms in a quasi-field.*

Let F be a quasi-field, B_1 and B_2 nonsingular hermitian matrices of order $n-1$ in F, and let a be a nonzero scalar. Let there be given a transformation of $\bar{x}_0ax_0+\bar{x}'B_1x$ into $\bar{x}_0ax_0+\bar{x}'B_2x$. Then explicit transformations are constructed which replace B_1 by B_2. This is an extension of a similar result due to Witt for fields. (Received July 23, 1945.)

ANALYSIS

154. E. F. Beckenbach: *On a characteristic property of linear functions.*

Let there be given a class of real functions $\{f(x)\}$, defined and continuous in a closed and bounded interval, such that there is a unique member of the family which, at arbitrary distinct x_1, x_2 in the interval, takes on arbitrary values y_1, y_2 respectively. The class of linear functions is an example. It is shown that a real function $g(x)$, defined and continuous in the interval, is a member of $\{f(x)\}$ if and only if for each x_0 interior to the interval there exists an $h_0=h_0(x_0)$ with $x_0 \pm h_0$ in the interval such that the member of $\{f(x)\}$ coinciding with $g(x)$ at $x_0 \pm h_0$ coincides with $g(x)$ also at x_0. (Received June 21, 1945.)

155. Stefan Bergman: *Pseudo harmonic vectors and their properties.*

The author applies the operator $\mathfrak{P}(f, \mathfrak{Q}, \mathfrak{T})$ introduced in Bull. Amer. Math. Soc. (vol. 49 (1943) p. 164) to complex functions $f=s^{(1)}(x, y)+is^{(2)}(x, y)$, for which $s_y^{(1)}=s_x^{(2)}$, $s_x^{(1)}=-s_y^{(2)}l(x)$ holds. Here $s_x^{(1)}=(\partial s^{(1)}/\partial x), \cdots$ and $l(x)$ is an analytic function of a real variable x. $\mathfrak{P}(s^{(1)}+is^{(2)}, \mathfrak{Q}, \mathfrak{T})$ yields a three-dimensional vector $\mathfrak{S}(X, Y, Z)=\mathfrak{S}^{(1)}+i\mathfrak{S}^{(2)}=\sum_{k=1}^{3}(S^{(k1)}+iS^{(k2)})\mathfrak{i}_k$ for which curl $\mathfrak{S}^{(1)}=0$, $S_X^{(11)}$

$+[S_Y^{(21)}+S_Z^{(31)}]l(x)=0$ and $S_X^{(22)}=l(X)S_Y^{(12)}$, $S_X^{(32)}=l(X)S_Z^{(12)}$, $S_Y^{(32)}=S_Z^{(22)}$, div $\mathfrak{S}^{(2)}=0$ holds. $S_X^{(11)}=(\partial S^{(11)}/\partial X), \cdots,$ $d\Omega(\mathfrak{S})=[S^{(11)}dX-S^{(21)}dY-S^{(31)}dZ]+i[l(X)S^{(12)}dX-S^{(22)}dY-S^{(32)}dZ]$ is a complete differential. Let $f=f(x, y; \zeta)$, $\zeta\in\mathfrak{Q}$, be a family of functions whose components satisfy the above system of equations. For every f there exists a subdivision of XYZ space into parts A_k, such that when $\mathfrak{X}=(X_0Y_0Z_0)\in A_k$ the operator $\mathfrak{P}(f, \mathfrak{Q}, \mathfrak{X})$ yields another vector $\mathfrak{S}_k$. However the $\mathfrak{S}_k$ are related to each other. Let $\mathfrak{J}=\sum_{k=1}^{n}\mathfrak{J}_k$, $\mathfrak{J}_k=\mathfrak{J}\cap A_k$, be a closed curve in the XYZ space. Then under suitable assumptions $\sum_{k=1}^{n}\int d\Omega(\mathfrak{S}_k)=(2\pi i)^{-1}\int \mathrm{Re}[f]d\zeta$, where the integral on the left is taken over $\mathfrak{J}_k$, the integral on the right over $\mathfrak{Q}$, and $\mathrm{Re}[f]$ denotes the sum of the "residues" of $f=f(x, y; \zeta)$ in a certain domain. (Received June 6, 1945.)

156. Arnold Dresden: *An extension of the equation of Lagrange,* $y=a+xf(y)$. Preliminary report.

In this paper the equation $F(y)+xG(y)=0$ is considered. Under suitable conditions on the functions F and G, a solution $y=\phi(x)$ exists, which assumes the values $y_0=\phi(0)$, where y_0 is a fixed root of $F(y)=0$, and which possesses derivatives of all orders in the vicinity of $x=0$. The general form of $\phi^{(n)}(0)$ is found in terms of derivatives of the functions F and G. A final explicit formula is conjectured, but not completely proved. Applications to the solution of algebraic equations, in particular to trinomial equations (compare the papers of Mellin), are immediate. The method of attack is entirely elementary, it can be applied also to equations of the form $F(y)+x_1G_1(y)+\cdots+x_kG_k(y)=0$, in which $x_1, \cdots, x_k$ are independent parameters. (Received July 31, 1945.)

157. Evelyn Frank: *On the zeros of polynomials with complex coefficients.*

Let $P(z)=z^n+A_1z^{n-1}+A_2z^{n-2}+\cdots+A_n$, $A_j=p_j+iq_j$, and form $Q(z)=p_1z^{n-1}+iq_2z^{n-2}+p_3z^{n-3}+iq_4z^{n-4}+\cdots$. In general, Q/P has the continued fraction expansion $1/c_1z+b_1+1+1/c_2z+b_2+\cdots+1/c_nz+b_n$, where the c_j are real and the b_j are pure imaginary. If k of the c_j are positive and $(n-k)$ negative, then $P(z)$ has k zeros with negative real parts and $(n-k)$ with positive real parts. This extends a theorem of Hurwitz (*Werke,* vol. 2, p. 533 ff.) with a method analogous to that of Wall (Amer. Math. Monthly vol. 52 (1945) pp. 308–322) for polynomials with real coefficients. Simultaneously, determinants analogous to the Hurwitz determinants are obtained for $P(z)$ with complex coefficients. If the partial denominators do not have the above form, the algorithm can be modified in several ways to obtain the number k. Bounds for the moduli of the zeros of $P(z)$ are found by Wall's method for real coefficients. Simple methods are given for expanding rational functions into continued fractions. These facilitate the use of the above theorem. There are important applications in electrical networks (Cauer, Archiv für Elektrotechnik vol. 17 (1926) pp. 355–388). (Received July 28, 1945.)

158. Josephine M. Mitchell: *Some properties of solutions of partial differential equations given by their series development.*

Let $U=\sum_{m,n=0}^{\infty}D_{mn}z^m\bar{z}^n$ $(D_{mn}=\bar{D}_{nm},\ z=x+iy,\ \bar{z}=x-iy)$ be a real analytic function of the two real variables x and y. Let U satisfy a linear partial differential equation $L(U)\equiv U_{z\bar{z}}+2\ \mathrm{Re}\ [(\sum_{m,n=0}^{\infty}a_{mn}z^m\bar{z}^n)U_z]+(\sum_{m,n=0}^{\infty}c_{mn}z^m\bar{z}^n)U=0$ $(U_z=(\partial U/\partial x-i\partial U/\partial y)/2,\ U_{z\bar{z}}=\Delta U/4)$. The a_{mn} and c_{mn} $(m, n=0, 1, 2, \cdots)$

are given. Then U is completely determined by the subsequence $\{D_{m0}\}$ since the remaining D_{mn} can be calculated by means of the a_{mn} and c_{mn}. Using results on a class of complex solutions of L introduced by Bergman [Trans. Amer. Math. Soc. vol. 57 (1945) pp. 299–331] and certain of his methods, the author gives sufficient conditions in terms of the $\{D_{m0}\}$ and $\{a_{m0}\}$ that U be continuous on the circle C: $x^2+y^2=$constant. (These conditions depend neither on the remaining a_{mn} nor on the c_{mn}.) Sufficient conditions that U have a jump on C as well as the size of the jump are given in terms of the same subsequences. These results are extended to other closed curves. By means of a theorem of Bergman [Duke Math. J. vol. 11 (1944) pp. 617–649] similar results are obtained for functions which satisfy differential equations of the fourth order. (Received July 28, 1945.)

159. A. F. Moursund: *Non-summability of the conjugate series of the Fourier series.*

In this paper it is shown that at a point x where $\int_0^t|f(x+u)-f(x-u)|du=O(t)$, as $t\to 0$, the divergence to $+\infty$ $[-\infty]$ of the Cauchy integral $\int_0^\pi[f(x+u)-f(x-u)]\cot u/2\,du$ is a necessary and sufficient condition for the divergence in the same sense of the conjugate Fourier series $\sum_{n=1}^\infty(b_n\cos nx-a_n\sin nx)$. (Received July 30, 1945.)

160. M. E. Munroe: *A note on weak differentiability of Pettis integrals.*

An example is given of a function defined to the space C; the function is Pettis integrable, but its integral fails to be weakly differentiable on a set of positive measure. A method is given for the specific construction of such a function to the space of continuous functionals over any compact metric space containing a homeomorph of the Cantor set. (Received July 13, 1945.)

161. Raphael Salem and Antoni Zygmund: *Lacunary power series and Peano curves.*

Let $f(z)=\sum_{k=1}^\infty a_k z^{n_k}$ be a lacunary power series of radius of convergence equal to 1, with $n_{k+1}/n_k\geqq\lambda>1$, and $\sum|a_k|<\infty$. If λ is larger than a certain absolute constant, and if the convergence of $\sum|a_k|$ is slow enough, the boundary values $f(e^{i\theta})$ fill completely a certain square. (Received July 17, 1945.)

162. Robert Schatten: *The cross-space of linear transformations.*

Let $\mathfrak{B}_1$, $\mathfrak{B}_2$ denote two Banach spaces, and $\mathfrak{B}_1^*$, $\mathfrak{B}_2^*$ their conjugate spaces. The direct product $\mathfrak{B}_1\otimes_\alpha\mathfrak{B}_2$ (Trans. Amer. Math. Soc. vol. 53 (1943) pp. 195–217) depending naturally on the norm α determines uniquely an "associate space" $\mathfrak{B}_1^*\otimes_{\alpha'}\mathfrak{B}_2^*$ (α' denotes the norm associated with α) and a conjugate space $(\mathfrak{B}_1\otimes_\alpha\mathfrak{B}_2)^*$. Let γ denote the greatest crossnorm, and λ the least crossnorm (whose associate is also a crossnorm). Then $\gamma'=\lambda$. Furthermore, $(\mathfrak{B}_1\otimes_\gamma\mathfrak{B}_2)^*$ is the Banach space of all linear transformations from $\mathfrak{B}_1$ into $\mathfrak{B}_2^*$ (from $\mathfrak{B}_2$ into $\mathfrak{B}_1^*$), while $\mathfrak{B}_1^*\otimes_\lambda\mathfrak{B}_2^*$ is characterized as the Banach space of all linear transformations from $\mathfrak{B}_1$ into $\mathfrak{B}_2^*$ (from $\mathfrak{B}_2$ into $\mathfrak{B}_1^*$), which may be approximated in norm by a sequence of linear transformations with finite-dimensional ranges. It is shown further, that for any reflexive crossnorm $\alpha(\alpha''=\alpha)$, the following statements are equivalent: (1) $\mathfrak{B}_1\otimes_\alpha\mathfrak{B}_2$ is reflexive, (2) $\mathfrak{B}_1^*\otimes_{\alpha'}\mathfrak{B}_2^*$ is reflexive, (3) $(\mathfrak{B}_1\otimes_\alpha\mathfrak{B}_2)^*=\mathfrak{B}_1^*\otimes_{\alpha'}\mathfrak{B}_2^*$ and $(\mathfrak{B}_1^*\otimes_{\alpha'}\mathfrak{B}_2^*)^*=\mathfrak{B}_1\otimes_\alpha\mathfrak{B}_2$. (Received May 29, 1945.)

163. C. F. Stephens: *Systems of linear difference equations containing a parameter.*

The author considers the system of equations $y_i(x+1)=\sum_{j=1}^{n}h_{ij}(x, \lambda)y_j(x)+g_i(x, \lambda)$, where $h_{ij}(x, \lambda)$ and $g_i(x, \lambda)$ are analytic in λ and continuous in x for $|\lambda| \leqq r$ and $|x| \geqq R$. One considers three cases: (1) $h_{ij}(x, 0) \not\equiv 0$, $g_i(x, \lambda) \equiv 0$; (2) $h_{ij}(x, 0) \not\equiv 0$, $g_i(x, 0) \equiv 0$; (3) $h_{ij}(x, 0) \equiv 0$, $g_i(x, 0) \equiv 0$. In cases (1) and (2) it is shown that there exist many solutions and in case (3) a unique solution. These solution functions are continuous functions of x and analytic functions of λ. The problems considered are shown to be special cases of more general problems treated in the author's doctoral thesis. (Bull. Amer. Math. Soc. abstract 50-5-148). (Received July 12, 1945.)

164. W. J. Trjitzinsky: *Theory of functions of intervals and applications to functions of a complex variable.*

This work generalizes theorems of Ridder and Besicovitch, relating to conditions under which an additive function of intervals is the integral of a suitably defined derivative of this function. These generalizations, together with certain other considerations, based on the author's earlier work in this field, lead to conditions for analyticity, as well as to a number of theorems on representation of certain general (non-analytic) classes of functions of a complex variable in terms of "double Cauchy integrals." (Received July 5, 1945.)

165. S. M. Ulam and John von Neumann: *Random ergodic theorems.*

The strong ergodic theorems are generalized from the case of a strictly determined flow to a process consisting of a combination of deterministic and random processes. The generalization may be illustrated by the following theorem constituting a special case: Let $S(p)$, $T(p)$ be two given but arbitrary measure preserving transformations of a measure space E into itself. Form all the combinations of the transformations: $S, T, S(S), S(T), T(S), T(T), S(S(T)), S(T(S)), \cdots$. The ergodic limit exists then for almost every point p of E and almost every choice of the infinite sequence obtained by applying S and T in turn at random, for example, $S(p), T(S(p)), T(T(S(p))), \cdots$. The metric transitivity of a given pair (or a given finite number) of transformations is studied and established for almost every pair of transformations in various special cases. These include the case where E is a compact connected group and S, T are, say, left translations (by elements s, t of the group E). (Received August 1, 1945.)

166. C. W. Vickery: *Concerning the notion of measure in metric spaces.*

The author has obtained two treatments of measure applicable to any metric space: (1) a generalization of the notion of measure of Jordan; (2) a generalization of the notion of measure of Borel-Lebesgue. Properties of spaces D^α, metric spaces of uncountably many dimensions, previously defined by the author (Bull. Amer. Math. Soc. vol. 45 (1939) pp. 456–462) are employed. Using treatment (2) the notion of a generalized Lebesgue-Stieltjes integral, applicable to real-valued functions defined on any metric space, is defined and properties of such an integral are studied. (Received June 21, 1945.)

167. H. S. Wall: *Analytic functions with positive real parts.*

Let $g_1, g_2, g_3, \cdots$ be any complex numbers such that $|g_p-(1/2)|<1/2$,

$p=1, 2, 3, \cdots$, and put $G_p=|g_p|^2/R(g_p)$, $\tau_p=I(g_p)/R(g_p)$. Let W denote the open region exterior to the cut along the real axis from -1 to $-\infty$, and let $(1+z)^{1/2}$ be the branch of the square root which is 1 for $z=0$. The continued fraction $1/1+G_1z/(1-i\tau_1(1+z)^{1/2})+(1-G_1)G_2z/(1-i\tau_2(1+z)^{1/2})+(1-G_2)G_3z/(1-i\tau_3(1+z)^{1/2})$ $+\cdots$ converges uniformly over every bounded closed region in W. The class of functions $F(z)$ which are analytic and have positive real parts in W, and equal 1 for $z=0$, is coextensive with the class of functions $(1+z)^{1/2}f(z)$, where $f(z)$ is the value of a continued fraction of the above form, or of a terminating continued fraction of that form in which the last G_p may equal 1. (Received July 12, 1945.)

168. H. S. Wall: *Theorems on arbitrary J-fractions.*

Let $1/(b_1+z)-a_1^2/(b_2+z)-a_2^2/(b_3+z)-\cdots$ $(a_p\neq 0)$ be an arbitrary J-fraction. Let $x_p=X_p(z)$, $x_p=Y_p(z)$ be the solutions of the system $-a_{p-1}x_{p-1}+(b_p+z)x_p-a_px_{p+1}=0$, $p=1, 2, 3, \cdots$ $(a_0=1)$ under the initial conditions $x_0=-1$, $x_1=0$ and $x_0=0$, $x_1=1$, respectively. The *indeterminate case* or the *determinate case* holds according as both the series $\sum|X_p(0)|^2$, $\sum|Y_p(0)|^2$ converge or at least one diverges, respectively. It is shown that in the indeterminate case, if the J-fraction converges for a single value of z, it converges for every value of z to a meromorphic function. If the J-fraction is *positive definite*, the associated J-matrix has one or infinitely many bounded reciprocals for $I(z)>0$ according as the determinate or the indeterminate case holds, respectively. Let $k_1, k_2, k_3, \cdots$ be numbers different from zero such that $\sum|k_{2p+1}|$ and $\sum|k_{2p+1}(k_2+k_4+\cdots+k_{2p})^2|$ converge. If $\lim_{p=\infty}|k_2+k_4+\cdots+k_{2p}|=\infty$, the continued fraction $1/k_1z+1/k_2+1/k_3z+1/k_4+\cdots$ converges for every z to a meromorphic function or else to the constant ∞. If the above limit does not exist, or is finite, then the continued fraction diverges by oscillation for every z. (Received June 8, 1945.)

Applied Mathematics

169. Stefan Bergman: *The integration of equations of fluid dynamics in the three-dimensional case.*

The author describes methods for the determination of potentials of *three-dimensional* flow patterns which are of interest in the theory of turbines. In order to obtain an approximate potential of an axially symmetric flow of a given type defined in the domain D, he determines the complex potential $g(z)$, $z=x+iy$, of a two-dimensional flow in the meridian plane of D, that is, in the region which is the intersection of D with the plane $\phi=$const. (r, θ, ϕ are the polar coordinates). Applying to $g(z)$ the operator introduced in Math. Zeit. vol. 24 (1926) pp. 641–669 he obtains a function which approximates the potential of the desired flow. (See, for example, the above paper, p. 655, where the potential of an axially symmetric flow in a turbine is given.) Using more complicated processes, potentials of general (not necessarily axis symmetric) flows can be obtained. These potential functions are used as first approximations to solutions of nonlinear equations of fluid dynamics. (Received July 30, 1945.)

170. Herbert Jehle: *A new approach to stellar statistics.*

A wave equation of the Schroedinger type, $[(\bar{\sigma}^2/2)\nabla^2+(\bar{\sigma}/i)\partial/\partial t-c^2-U]\cdot\{|\psi|\exp(iS/\bar{\sigma})\}=0$, where $\bar{\sigma}$ is a constant, c the velocity of light and U the potential field, is known to admit a hydrodynamical interpretation: It splits into two real