ON SURFACES OF CLASS K_1

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The purpose of this note is to make a comment bearing upon the remarkable results of Radó on the semi-continuity of double integrals in parametric form.[1]

The essence of the situation, without at first attempting precision, is this. A continuous surface is of class K_1 if and only if it has a representation for which the classical double integral area formula has meaning. (It is understood that the integration is in the sense of Lebesgue.) This class of continuous surfaces is variously employed in Radó's paper. The primary object of this note is to show that *every surface is of class* K_1.

In putting things more precisely it is both convenient and economical to treat the matter in its true light; namely, as a corollary to Radó's paper. Thus the notation and terminology here follow that of Radó, and numbers in parentheses refer to the appropriate paragraphs in his paper.

To preserve a certain measure of continuity a few of the salient concepts are here reviewed.

A *continuous surface* S (1.21) is, by definition, an equivalence class of *triples of continuous functions* (1.6). If the definition of the equivalence relation is strengthened by the addition of an *orientation* requirement, then any one of the resulting equivalence classes is known as an *oriented continuous surface* ${}_oS$ (1.23). In each case any triple in the equivalence class is known as a *representation* of the surface.

The notation (T, B) is used generically to denote a *continuous triple of functions*.

$$T: \quad x^i(u^1, u^2), \qquad (u^1, u^2) \in B, \ i = 1, 2, 3,$$

where B is the closure of some Jordan region in the plane. The abbreviated notation

$$T: \quad x(u), \qquad u \in B,$$

is also employed (1.6).

If a continuous triple (T, B) is such that the six partial derivatives exist almost everywhere in B^0, then the Jacobians are denoted by

$$X^1(u^1, u^2) = \frac{\partial(x^2, x^3)}{\partial(u^1, u^2)}, \qquad X^2(u^1, u^2) = \frac{\partial(x^3, x^1)}{\partial(u^1, u^2)},$$

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[1] Tibor Radó, *On the semi-continuity of double integrals in parametric form*, Trans. Amer. Math. Soc. vol. 51 (1942) pp. 336–361.

$$X^3(u^1, u^2) = \frac{\partial(x^1, x^2)}{\partial(u^1, u^2)}$$

and, of course, also exist almost everywhere in B^0.

If, in addition, the Jacobians are *summable* then the *continuous triple* (T, B) is said to be of *class* K_1 (1.8). (It is to be noted that this class K_1 refers to continuous triples and *not* to continuous surfaces.)

A continuous surface S (*oriented continuous surface* ${}_oS$) is said to be of *class* K_1 in the event it has a representation which is of *class* K_1 (1.26).

The whole matter can be viewed from another point of view. In the classical theory of area, the term "area" is applied to what is here called a continuous triple (T, B), and the well known expression

$$\iint_{B^0} [(X^1)^2 + (X^2)^2 + (X^3)^2]^{1/2} du^1 du^2$$

is employed. It is obvious that the classical discussion applies only to those continuous triples for which the radical $[(X^1)^2+(X^2)^2+(X^3)^2]^{1/2}$ exists, and is in some sense integrable. In the event the *integration* is understood to be that of *Lebesgue*, however, the classical area formula above can be applied to *all continuous triples of class* K_1. Indeed, the continuous triples of class K_1 are precisely those for which the expression

$$\|X\| = [(X^1)^2 + (X^2)^2 + (X^3)^2]^{1/2}$$

is summable over B^0.

If we represent the classical area formula by $I(T, B, \|X\|)$, the statement can be made in the following form:

A continuous triple (T, B) *is of class* K_1 *if and only if* $I(T, B, \|X\|)$ *exists.*

It is now possible to state the theorem.

THEOREM. *Every continuous surface, whether oriented or not, is of class* K_1. *Alternately, every continuous surface* S (*oriented continuous surface* ${}_oS$) *has a representation* (T, B) *such that* $I(T, B, \|X\|)$ *exists.*

PROOF. Suppose S (${}_oS$) is a continuous surface (oriented continuous surface) with a representation

$$\bar{T}: \quad \bar{x}(\bar{u}), \qquad \bar{u} \in \bar{B}.$$

There is clearly no lack of generality in assuming that $\bar{B}$ is the closed unit square $0 \leqq \bar{u}^1 \leqq 1$, $0 \leqq \bar{u}^2 \leqq 1$.

To prove the theorem a representation

$$T: \quad x(u), \qquad u \in B,$$

of S $({}_oS)$ must be found which is of class K_1.

Suppose $\gamma(t)$ is the familiar Cantor function. The properties of this function here employed are: $d\gamma/dt=0$ on the open set of unit length which is complementary to the Cantor set, $\gamma(0)=0$, $\gamma(1)=1$.

Consider the triple (T, B) where

$$B: \quad 0 \leqq u^1 \leqq 1, \qquad 0 \leqq u^2 \leqq 1.$$

$$T: \quad x^i(u^1, u^2) = \bar{x}^i(\gamma(u^1), \gamma(u^2)), \qquad i = 1, 2, 3.$$

If we define the transformation $\bar{u}=\mu(u)$ by the couple

$$\bar{u}^1 = \gamma(u^1), \qquad \bar{u}^2 = \gamma(u^2)$$

the triple can also be defined in condensed form by

$$T: \quad x(u) = \bar{x}(\mu(u)), \qquad u \in B.$$

It is easy to see that

$$\frac{\partial x^i}{\partial u^1} = 0 = \frac{\partial x^i}{\partial u^2}, \qquad i = 1, 2, 3,$$

almost everywhere in B^0. Therefore the Jacobian

$$X^i(u^1, u^2) = 0, \qquad i = 1, 2, 3,$$

almost everywhere in B^0. Thus $I(T, B, \|X\|)$ exists, and in point of fact is equal to zero.

It remains but to show that (T, B) is, indeed, a representation of S $({}_oS)$. In other words it must be shown that for any $\epsilon>0$ there is a topological transformation

$$\bar{u} = \tau_\epsilon(u)$$

such that the *distance*

$$\|\bar{x}(\tau_\epsilon(u)) - x(u)\| = \|\bar{x}(\tau_\epsilon(u)) - \bar{x}(\mu(u))\| < \epsilon, \qquad u \in B.$$

The required transformation $\bar{u}=\tau_\epsilon(u)$ is selected in the following fashion:

Since $\bar{x}(\bar{u})$ is a continuous triple there is a $\delta>0$ such that the distance $\|\bar{x}(\bar{u})-\bar{x}(\bar{v})\|<\epsilon$ for $\|\bar{u}-\bar{v}\|<\delta$.

Let $\gamma^*(t)$ be a continuous and strictly monotone function such that

$$\gamma^*(0) = 0, \quad \gamma^*(1) = 1, \quad \|\gamma^*(t) - \gamma(t)\| < \delta/2 \text{ for } 0 \leqq t \leqq 1.$$

Now consider the transformation $\bar{u}=\tau_\epsilon(u)$ defined by the couple

$$\bar{u}^1 = \gamma^*(u^1), \qquad \bar{u}^2 = \gamma^*(u^2).$$

This transformation is topological, since γ^* is strictly monotone in addition to being continuous, and

$$\|\tau_\epsilon(u) - \mu(u)\| < \delta, \qquad u \in B.$$

Hence

$$\|\bar{x}(\tau_\epsilon(u)) - \bar{x}(\mu(u))\| < \epsilon, \qquad u \in B.$$

On observing that τ_ϵ is *sense preserving* (1.18) the proof is complete.

It has been noted that the special representation (T, B) has the property that the Jacobians vanish almost everywhere in B^0. This proves the following corollary.

COROLLARY. *Every surface S $({}_oS)$ has a representation (T, B) for which B is the closed unit square, and, in addition,* $I(T, B, \|X\|) = 0$.

In a sense this shows the complete unreliability of the classical area formula if it is applied at random to any representation of a continuous surface (oriented continuous surface) for which the formula has meaning—there is always a representation yielding the value zero. As a matter of fact, two of the most elegant theorems of Radó concern a criterion by which it is possible to select precisely those representations of a continuous surface (oriented continuous surface), if any, to which the classical formula can be applied so as to obtain the Lebesgue area (3.14, 3.18). They are here condensed to read as follows:

THEOREM (RADÓ). *If S $({}_oS)$ is a continuous surface (oriented continuous surface) of class K_1, and (T, B) is a representation of S $({}_oS)$ of class K_1, then the Lebesgue area*

$$L(S) \geqq I(T, B, \|X\|) \qquad (L({}_oS) \geqq I(T, B, \|X\|)).$$

The sign of equality holds if and only if (T, B) is of class K_2 (1.19).

In virtue of the remarks in this note the statement can now be somewhat strengthened.

THEOREM (RADÓ). *Any continuous surface S (oriented continuous surface ${}_oS$) has a representation (T, B) of class K_1 and*

$$L(S) \geqq I(T, B)\|X\|.) \qquad (L({}_oS) \geqq I(T, B, \|X\|)).$$

The sign of equality holds if and only if (T, B) is of class K_2.

Several questions arise as a natural consequence of the fact that every surface has a representation for which the classical area is zero. Though these questions will have to await somewhat quieter times for personal investigation they are presented here in the hopes that they will interest others.

QUESTION 1. *Are those representations* (T, B) *of* S $({}_oS)$ *for which* $I(T, B, \|X\|)=0$ *of the second category in the collection of all representations of* S $({}_oS)$ *which are of class* K_1?

For any continuous surface S (oriented continuous surface ${}_oS$) let:
$I^*(S)=\sup\ I(T, B, \|X\|)$ for all representations (T, B) of S which are of class K_1.
$(I^*({}_oS)=\sup I(T, B, \|X\|)$ for all representations (T, B) of ${}_oS$ which are of class K_1.)
The theorem of Radó guarantees that

$$L(S) \geqq I^*(S) \qquad (L({}_oS) \geqq I^*({}_oS)).$$

QUESTION 2. *Does* $I^*(S)$ $(I^*({}_oS))$ *share an important property of the Lebesgue integral; explicitly, is* $I^*(S)$ $(I^*({}_oS))$ *a lower semi-continuous function of* S $({}_oS)$?

QUESTION 3. *Is* $I^*(S)=L(S)$ $(I^*({}_oS)=L({}_oS))$?

An affirmative answer to the last question would be of compelling interest.

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