If \( p = q + 1 \), replace (21) by
\[
[(2\alpha_k - \beta_k)(1 - x) + (A - B)x]F
= \alpha_k(1 - x)F(\alpha_k +) + (\alpha_k - \beta_k)F(\alpha_k -)
- x \sum_{j=1}^{k-1} V_j, iF(\beta_j +); \quad k = 1, 2, \ldots, p.
\]

ON THE GROWTH OF THE SOLUTIONS OF ORDINARY
DIFFERENTIAL EQUATIONS

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In a recent paper,\(^1\) N. Levinson gave four theorems concerning the
behaviour of the solutions of the differential equation of elastic vibrations
\[
\frac{d^2x}{dt^2} + \phi(t)x = 0
\]
as \( t \to +\infty \). It is the purpose of this note to give generalizations of the
Theorems I and III of Levinson by making use of certain inequalities
concerning homogeneous equations of the first order

\[
\frac{dx_i}{dt} + \sum_{k=1}^{n} a_{ik}x_k = 0, \quad i = 1, \ldots, n.
\]

Theorems I and III of Levinson run as follows:

**Theorem I.** If \( \alpha(t) \) denotes the integral

\[
\alpha(t) = \int_0^t \left| \phi(t) - c^2 \right| dt,
\]
then

\[
x(t) = O\{\exp (\alpha(t)/2c)\}.
\]

**Theorem III.** If \( \alpha(t) \) is \( O(t) \) then

\[
\lim_{t \to \infty} \sup \left| x(t) \exp (\alpha(t)/2c) \right| > 0.
\]


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The Polish mathematician Z. Butlewski in a paper written in Polish\footnote{O całkach rzeczywistych równań różniczkowych zwyczajnych, Wiadomości Matematyczne vol. 44 (1937) pp. 17–81.} sets

\begin{equation}
    r = \left( \sum_{i=1}^{n} x_{i}^{2} \right)^{1/2},
\end{equation}
\begin{equation}
    \phi(t) = \sum_{i}^{n} a_{ii} \left( \frac{x_{i}}{r} \right)^{2} + \sum_{i,k=1}^{n} (a_{ik} + a_{ki}) \frac{x_{i}x_{k}}{r^{2}},
\end{equation}
where \( i \neq k \), \( i < k \), and obtains immediately from (2)
\begin{equation}
    r = C \exp \left( - \int_{t_{0}}^{t} \phi(\tau) d\tau \right).
\end{equation}

Setting
\begin{equation}
    \alpha_{ii} = \int_{t_{0}}^{t} | a_{ii} | d\tau, \quad \beta_{ij} = \int_{t_{0}}^{t} | a_{ij} + a_{ji} | d\tau,
\end{equation}
we have
\begin{equation}
    r \leq C \exp \left( \sum_{i=1}^{n} \alpha_{ii} + \frac{1}{2} \sum_{i,j=1; i < j}^{n} \beta_{ij} \right)
\end{equation}
and from this we have the following theorem.

**Theorem XVIII of Butlewski.** If \( \alpha_{ii} < + \infty \) and \( \beta_{ij} < + \infty \), all the systems \( x_{i}, i = 1, \ldots, n \), of solutions of (2) are bounded.

With the designation
\begin{equation}
    M(t) = \max \int_{t_{0}}^{t} | a_{ij} | d\tau,
\end{equation}
we have
\begin{equation}
    r \leq C \exp \left( n M(t) \right).
\end{equation}

We can complete Butlewski's theorem by remarking that
\begin{equation}
    r \geq C \exp \left( - \sum_{i}^{n} \alpha_{ii} - \frac{1}{2} \sum_{i,j=1; i < j}^{n} \beta_{ij} \right),
\end{equation}
and
\begin{equation}
    r \geq C \exp \left( - n M(t) \right).
\end{equation}
In the particular case $n = 2$ Butlewski introduces polar coordinates

$$x_1 = \rho \cos \phi, \quad x_2 = \rho \sin \phi$$

and obtains

$$\rho = C \exp \left( - \int_{t_0}^{t} \left\{ a_{11} \cos^2 \phi + (a_{12} + a_{21}) \sin \phi \cos \phi + a_{22} \sin^2 \phi \right\} \, dt \right).$$

The maximum of $F(\phi) = - \left\{ a_{11} \cos^2 \phi + (a_{12} + a_{21}) \sin \phi \cos \phi + a_{22} \sin^2 \phi \right\}$ is

$$2^{-1} \left\{ - a_{11} - a_{22} + ((a_{11} - a_{22})^2 + (a_{12} + a_{21})^2)^{1/2} \right\},$$

so that

$$\rho \leq C \exp \left( \frac{1}{2} \int_{t_0}^{t} \left\{ - a_{11} - a_{22} + ((a_{11} - a_{22})^2 + (a_{12} + a_{21})^2)^{1/2} \right\} \, dt \right).$$

Thus we obtain the following theorem.

**Theorem XIX of Butlewski.** $x_1$, $x_2$ are limited if

$$\frac{1}{2} \int_{t_0}^{t} \left\{ - a_{11} - a_{22} + ((a_{11} - a_{22})^2 + (a_{12} + a_{21})^2)^{1/2} \right\} \, dt$$

is limited.

In the same manner we obtain

$$\rho \geq C \exp \left( \frac{1}{2} \int_{t_0}^{t} \left\{ - a_{11} - a_{22} - ((a_{11} - a_{22})^2 + (a_{12} + a_{21})^2)^{1/2} \right\} \, dt \right),$$

completing Butlewski’s results.

The linear differential equation

$$x'' + \psi(t)x' + \phi(t)x = 0$$

with continuous $\phi$ and $\psi$ can be transformed into a system (2) by setting

$$x' = \lambda x_2, \quad x = x_1,$$
\( \lambda \neq 0 \), continuous in \( t \). We obtain the system

\[
\begin{align*}
\frac{dx_1}{dt} - \lambda x_2 &= 0, \\
\frac{dx_2}{dt} + \frac{\phi}{\lambda} x_1 + \left( \frac{\lambda'}{\lambda} + \psi \right) x_2 &= 0,
\end{align*}
\]

\( a_{11} = 0, a_{12} = -\lambda, a_{21} = \phi/\lambda, a_{22} = \lambda'/\lambda + \psi. \)

The inequalities (17), (19) now take the form

\[
\rho \leq C \exp \left( \frac{1}{2} \int_{t_0}^{t} \left\{ - \frac{\lambda'}{\lambda} - \psi + \left( \frac{\lambda'}{\lambda} + \psi \right)^2 \right\} d\tau \right),
\]

\[
\rho \geq C \exp \left( \frac{1}{2} \int_{t_0}^{t} \left\{ - \frac{\lambda'}{\lambda} - \psi - \left( \frac{\lambda'}{\lambda} + \psi \right)^2 \right\} d\tau \right).
\]

Taking \( \lambda = c \), Butlewski obtains

\[
\rho \leq C \exp \left( \frac{1}{2} \int_{t_0}^{t} \left\{ - \psi + \left( \psi^2 + \left( \frac{\phi}{c} - c \right)^2 \right)^{1/2} \right\} d\tau \right),
\]

and we can add the inequality

\[
\rho \geq C \exp \left( \frac{1}{2} \int_{t_0}^{t} \left\{ - \psi - \left( \psi^2 + \left( \frac{\phi}{c} - c \right)^2 \right)^{1/2} \right\} d\tau \right).
\]

We shall now generalize Levinson's Theorem III for the equation (20).

We can suppose \( x > 0, x' < 0 \), otherwise we should have an infinite number of values \( t = t_i, i = 1, 2, \ldots ; t_i \to \infty \), with

\[
|x(t)| \geq C \exp \left( \frac{1}{2} \int_{t_0}^{t} \left\{ - \psi + \left( \psi^2 + \left( \frac{\phi}{c} - c \right)^2 \right)^{1/2} \right\} d\tau \right).
\]

Consider the intervals \( n \leq t \leq n+1 \). We have

\[
x(n + 1) - x(n) = \int_{n}^{n+1} x'(t) dt,
\]

and denoting by \( x'_n \) the maximum of \( x'(t) \) in \( \langle n, n+1 \rangle \),

\[
x(n) \geq - x'_n, \quad x'_n = x'(t_n), \quad n \leq t_n \leq n + 1.
\]

We have
\[ x'(n) - x_n' = \int_{t_n}^{t} x''(t) dt = - \int_{t_n}^{t} (\phi x + \psi x') dt, \]

\[ |x'(n)| \leq x(n) + x(n) \int_{n}^{n+1} |\phi| dt + |\psi|_{\text{max}} x(n) \]

\[ = x(n) \left\{ 1 + \int_{n}^{n+1} |\phi| dt + |\psi|_{\text{max}} \right\}. \]

\(|\psi|_{\text{max}}\) is the maximum of \(|\psi|\) in \(n \leq t \leq n+1\).

We obtain

\[ C \exp \left( \frac{1}{2} \int_{t_0}^{t} \left\{ -\psi - \left( \psi^2 + \left( \frac{\phi}{c} - c \right)^2 \right)^{1/2} \right\} d\tau \right) \]

\[ \begin{array}{c}
     1 + \frac{1}{c^2} \left( 1 + |\psi|_{\text{max}} + \int_{n}^{n+1} |\phi| dt \right)
   \end{array} \]

The result is the following theorem.

Theorem. There exist infinite values of \(t = t_i, t_i \to \infty\), for which we have (27) if the following conditions are satisfied:

1. \(\alpha(t)\) is of order \(O(t)\).
2. \(|\psi|\) is bounded.

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