A NOTE ON SYSTEMS OF HOMOGENEOUS ALGEBRAIC EQUATIONS

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1. Introduction. Consider a system of algebraic equations

\[ f_1(x_1, x_2, \ldots, x_n) = 0, \]
\[ f_2(x_1, x_2, \ldots, x_n) = 0, \]
\[ \ldots \ldots \ldots \ldots \]
\[ f_h(x_1, x_2, \ldots, x_n) = 0, \]

where \( f_i \) is a homogeneous polynomial of degree \( r_i \) with coefficients belonging to a given field \( K \). We interpret \( x_1, x_2, \ldots, x_n \) as homogeneous coordinates in an \((n-1)\)-dimensional projective space. When \( n > h \), the system (1) has non-trivial solutions \((x_1, x_2, \ldots, x_n)\) in an algebraically closed extension field of \( K \), but there may not exist any such solutions in \( K \) itself. It is, in general, extremely difficult to decide whether adjunction of irrationalities of a certain type to \( K \) is sufficient to guarantee the existence of non-trivial solutions of (1) in the extended field. However, the situation is much simpler, when \( n \) is very large, in the sense that \( n \) lies above a certain expression depending on the number of equations \( h \) and the degrees \( r_1, r_2, \ldots, r_h \).

We shall show:

**Theorem A.** For any system of \( h \) positive degrees \( r_1, r_2, \ldots, r_h \) there exists an integer \( \Phi(r_1, r_2, \ldots, r_h) \) such that for \( n \geq \Phi(r_1, r_2, \ldots, r_h) \) the system (1) has a non-trivial solution in a soluble extension field \( K_1 \) of \( K \). The field \( K_1 \) may be chosen such that its degree \( N_1 \) over \( K \) lies below a value depending on \( r_1, r_2, \ldots, r_h \) alone and that any prime factor of \( N_1 \) is at most equal to \( \max(r_1, r_2, \ldots, r_h) \).

This Theorem A is evidently contained in the following theorem.

**Theorem B.** For any system of positive integers \( r_1, r_2, \ldots, r_h \) and any integer \( m \geq 0 \), there exists an integer \( \Phi(r_1, r_2, \ldots, r_h; m) \) with the following property: For \( n \geq \Phi(r_1, r_2, \ldots, r_h; m) \), there exists a soluble extension field \( K_2 \) of \( K \) such that all points \((x_1, x_2, \ldots, x_n)\) of an \( m \)-dimensional linear manifold \( L \), defined in \( K_2 \), satisfy the equations (1). Here \( K_2 \) may be chosen so that its degree \( N_2 \) over \( K \) lies below a bound depending on \( r_1, r_2, \ldots, r_h \) and \( m \) and that no prime factor of \( N_2 \) exceeds \( \max(r_1, r_2, \ldots, r_h) \).

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At the same time, we shall prove the theorem:

**Theorem C.** Assume that the field $K$ has the following property:

(*) For every integer $r > 0$, there exists an integer $\Psi(r)$ such that for $n \geq \Psi(r)$ every equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

with coefficients $a_i$ in $K$ has a non-trivial solution in $K$.

Then, for every system of positive degrees $r_1, r_2, \ldots, r_h$ and every integer $m \geq 0$, there exists an expression $\Omega(r_1, r_2, \ldots, r_h; m)$ with the following property: For $n \geq \Omega(r_1, r_2, \ldots, r_h; m)$, there exists an $m$-dimensional linear manifold $M$, defined in $K$, whose points satisfy the equations (1).

We shall prove Theorem C in §2. The changes necessary in order to obtain Theorem B are obvious. In §3, some applications are given. One of them is concerned with Hilbert's resolvent problem. We prove here a recent conjecture of B. Segre.1

2. **Proof of Theorem C.** 1. Assume that Theorem C is not true. We choose a system $r_1, r_2, \ldots, r_h; m$ for which no $\Omega(r_1, r_2, \ldots, r_h; m)$ exists. We select this system such that $\max (r_1, \ldots, r_h) = s$ has the smallest possible value, and that for fixed $s$ the number $h$ has the smallest possible value. If $r'_1, r'_2, \ldots, r'_h$ is any system of positive integers and $m'$ a non-negative integer, then $\Omega(r'_1, r'_2, \ldots, r'_h; m')$ exists, if either

$$\max (r'_1, r'_2, \ldots, r'_h) < s$$

or if

$$\max (r'_1, r'_2, \ldots, r'_h) = s, \quad h' < h.$$  

Assume first that $h > 1$. We may assume that $r_h = s$. It follows that $\Omega(r_1, r_2, \ldots, r_{h-1}; m)$ exists (cf. the conditions (3a) and (3b)) and also that $\Omega(s; m' - 1)$ exists for any integer $m' > 0$. We set $m' = \Omega(r_1, \ldots, r_{h-1}; m)$. If $n \geq \Omega(s; m' - 1)$, the equation $f_h = 0$ is satisfied by all points of an $(m' - 1)$-dimensional linear manifold $M_1$. If we restrict ourselves to points of $M_1$, we may express $x_1, \ldots, x_n$ linearly and homogeneously by $m'$ parameters $y_1, \ldots, y_{m'}$ with coefficients in $K$. Then $f_i(x_1, \ldots, x_n)$ becomes a homogeneous polynomial $g_i$ of $y_1, \ldots, y_{m'}$. The degree of $g_i$ is $r_i$; the coefficients of $g_i$ belong to $K$. In particular, $g_h$ vanishes identically. In order to solve

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1 B. Segre, Ann. of Math. vol. 46 (1945) p. 287. Added September 10: In the meantime, I learned from Mr. Segre that he also found Theorem A from which the proof of the conjecture can be derived.
(1), we have to solve

\[ g_1 = 0, g_2 = 0, \ldots, g_{h-1} = 0. \]

Since \( m' = \Omega(r_1, \ldots, r_{h-1}; m) \), the equations (4) will be satisfied by all points of an \( m \)-dimensional manifold \( M_2 \) of the \((y_1, \ldots, y_m')\)-space. This then gives an \( m \)-dimensional linear manifold of the \((x_1, \ldots, x_n)\)-space for which the equations (1) hold. But this shows that the expression \( \Omega(r_1, \ldots, r_h; m) \) exists; we may take

\[ \Omega(r_1, \ldots, r_h; m) = \Omega(\max (r_1, \ldots, r_h); \Omega(r_1, \ldots, r_{h-1}; m) - 1). \]

Hence the case \( h > 1 \) is impossible.

2. We now consider the case \( h = 1 \). The system (1) consists of only one equation

\[ f(x_1, x_2, \ldots, x_n) = 0 \]

of degree \( r_1 = s \).

From the way the number \( s \) was chosen it follows that \( \Omega(s; m) \) does not exist while for every system \( r'_1, r'_2, \ldots, r'_h \) with \( r'_1 < s, r'_2 < s, \ldots, r'_h < s \) and all \( m' \) the existence of \( \Omega(r'_1, r'_2, \ldots, r'_h; m') \) may be assumed.

We first discuss the case \( m = 0 \). Denoting the point \((x_1, x_2, \ldots, x_n)\) by \( \xi \), we write \( f(x_1, x_2, \ldots, x_n) = f(\xi) \).

If \( \xi_1, \xi_2, \ldots, \xi_n \) are \( n \) points whose coordinates are independent indeterminates and if \( u_1, u_2, \ldots, u_n \) are \( n \) further independent indeterminates, we may set

\[ f(u_1\xi_1 + u_2\xi_2 + \cdots + u_n\xi_n) = \sum u_1^\mu u_2^\nu \cdots u_n^\tau f_{\mu,\nu,\ldots,\tau}(\xi_1, \xi_2, \ldots, \xi_n), \]

where the sum on the right side extends over all systems of \( n \) non-negative integers \((\mu, \nu, \ldots, \tau)\) with

\[ \mu + \nu + \cdots + \tau = s. \]

The expressions \( f_{\mu,\nu,\ldots,\tau}(\xi_1, \xi_2, \ldots, \xi_n) \) (the polar forms of \( f \)) are homogeneous polynomials in the coordinates of each \( \xi_i \). As is easily seen, \( f_{\mu,\nu,\ldots,\tau}(\xi_1, \xi_2, \ldots, \xi_n) \) is of degree \( \mu \) in the coordinates of \( \xi_1 \), of degree \( \nu \) in the coordinates of \( \xi_2, \ldots, \xi_n \), of degree \( \tau \) in the coordinates of \( \xi_n \).

Let \( a_1 \neq 0 \) be a fixed point.\(^2\) Choose \( n - 1 \) points \( \xi_1, \xi_2, \ldots, \xi_{n-1} \) which together with \( a_1 \) form a full linearly independent system, and set \( \eta = y_1\xi_1 + y_2\xi_2 + \cdots + y_{n-1}\xi_{n-1} \) where the coefficients \( y_1, y_2, \ldots, y_{n-1} \) are indeterminates.

Consider the system of equations

\[^2\text{We denote by } \mathbf{0} \text{ the row } (0, 0, \ldots, 0) \text{ consisting of } n \text{ numbers } 0.\]
These equations are homogeneous in $y_1, y_2, \ldots, y_n-1$; the degrees are 1, 2, \ldots, $s-1$ respectively.

From the remarks above it follows that the expression $\Omega(1, 2, \ldots, s-1; 0)$ exists. Hence for sufficiently large $n$ the equations (6) will have a non-trivial solution. Let $\psi = a_2$ be the corresponding point $\psi$. Then $a_1$ and $a_2$ are linearly independent.

Let $\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_{n-2}$ be a system of points which together with $a_1$ and $a_2$ form a full linearly independent system and set

$$\zeta = z_1 \epsilon'_1 + \cdots + z_{n-2} \epsilon'_{n-2}$$

with indeterminate coefficients $z_1, z_2, \ldots, z_{n-2}$. Consider next the equations

$$(7) \quad f_{\mu, \nu, \rho, 0, \ldots, 0}(a_1, a_2, \zeta, o, \ldots, o) = 0,$$

where $\mu, \nu, \rho$ range over all systems of non-negative integers with

$$\mu + \nu + \rho = s, \quad 0 < \rho < s.$$ 

Again, $\Omega(r'_1, \ldots, r'_h; 0)$ exists for the degrees $r'_1, \ldots, r'_h$ of these equations in $z_1, z_2, \ldots, z_{n-2}$. It follows for sufficiently large $n$ that the system (7) has a non-trivial solution $(z_1, \ldots, z_{n-2})$. Let $z = a_3$ be the corresponding point. Then $a_1, a_2, a_3$ are linearly independent.

Set $t = \Psi(s)$.

Assuming that $n$ is sufficiently large we continue with our procedure until we obtain $t$ linearly independent points $a_1, a_2, \ldots, a_t$ such that

$$f_{\mu, \nu, \ldots, \tau}(a_1, a_2, \ldots, a_t, o, \ldots, o) = 0$$

for every system of $n$ non-negative indices $(\mu, \nu, \ldots, \tau)$ with $\mu + \nu + \cdots + \tau = s$ in which the first $t$ of our indices are all less than $s$.

For $\xi_1 = a_1, \xi_2 = a_2, \ldots, \xi_t = a_t, \xi_{t+1} = o, \ldots, \xi_n = o$, the identity

In part 2 of the proof we mean by "sufficiently large $n" all values of $n$ which lie above a suitable lower bound $\Lambda(s)$ depending only on $s$.

In the case of Theorem B, we take $t=2$. The equation (8) will have a solution if we extend the field $K$ by the adjunction of an $s$th root.

If one of the last $n-t$ indices in $(\mu, \nu, \ldots, \tau)$ does not vanish, this equation is trivial, since the left side then contains an $\xi_i = o$ to a positive degree.
(5) gives a relation
\[ f(u_1a_1 + u_2a_2 + \cdots + u_ia_i) = \sum_{i=1}^{t} a_iu_i, \]
where \( a_i \) is a certain number of \( K \). Actually, \( a_i = f(a_i) \).

Since \( t = \Psi(s) \), the equation
\[ \sum_{i=1}^{t} u_ia_i = 0 \]
has a non-trivial solution \( (u_1, u_2, \cdots, u_i) \) in \( K \). The corresponding point \( r = \sum u_ia_i \) then yields a non-trivial solution of the equation \( (r) = 0 \) in \( K \).

This argument shows the existence of \( \Omega(s; 0) \).

3. We assume that the existence of \( m' = \Omega(s; m - 1) \) has already been shown. If \( n \) is sufficiently large,, the result of 2 shows that we may find a point \( a_1 \neq 0 \) such that
\[ f(a_1) = 0. \]

Consider again the equations (6) where \( \eta \) has the old significance. Again, \( \Omega(1, 2, \cdots, s-1; m'-1) \) exists. If \( n \geq \Omega(1, 2, \cdots, s-1; m'-1) \), it follows that there exists an \((m'-1)\)-dimensional linear space \( M_0 \) such that the equations (6) hold for all points \( \eta \) of \( M_0 \), and that \( M_0 \) does not contain \( a_1 \).

The identity (5) for \( \xi_1 = a_1, \xi_2 = \eta, \xi_3 = 0, \cdots, \xi_n = 0 \) yields
\[ f(u_1a_1 + u_2\eta) = u_2f(\eta), \]
on account of (6) and (9). Restricting the point \( \eta \) to the linear manifold \( M_0 \), we may consider the coordinates of \( \eta \) as linear homogeneous functions of \( m' \) parameters \( z_1, z_2, \cdots, z_m' \). Since \( m' = \Omega(s; m - 1) \), it follows that there exists an \((m-1)\)-dimensional linear subspace \( M_1 \) of \( M_0 \) such that \( f(\eta) = 0 \) for all points \( \eta \) of \( M_1 \). But (10) shows that \( a_1 \) and \( M_1 \) together span an \( m \)-dimensional linear space \( M \) which consists entirely of solutions of \( f(r) = 0 \). This proves the existence of \( \Omega(s; m) \) which contradicts the assumptions made above.

This finishes the proof of Theorem C. The same method yields the proof of Theorem B, and hence the Theorem A.

3. Applications. Consider the general algebraic equation of degree

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6 In part 3 of the proof we shall say that \( n \) is sufficiently large if it lies above a suitable lower bound \( M(s, m) \), depending on \( s \) and \( m \) only.

7 For Hilbert's resolvent problem, see the paper by Segre quoted in footnote 1 and the literature mentioned in this paper, also A. Wiman, Nova Acta Uppsala (1927).
n in one unknown
\[ f(x) = x^n + a_1x^{n-1} + \cdots + a_n = 0. \]

If the roots are \( \omega_1, \omega_2, \cdots, \omega_n \) and if we set
\[ \theta_i = u_0 + u_1\omega_i + \cdots + u_{n-1}\omega_i^{n-1} \]
then the \( \theta_i \) are the roots of an equation
\[ g(x) = x^n + b_1x^{n-1} + \cdots + b_n = 0 \]
and it is well known that the coefficient \( b_i \) of this Tschirnhaus transformation is a homogeneous polynomial \( B_i(u_0, u_1, \cdots, u_{n-1}) \) of degree \( i \) in the \( u_0, u_1, \cdots, u_{n-1} \). For a fixed \( k \), we determine the quantities \( u_0, u_1, \cdots, u_{n-1} \) as a non-trivial solution of the equations
\[ B_1(u_0, u_1, \cdots, u_{n-1}) = 0, \]
\[ B_2(u_0, u_1, \cdots, u_{n-1}) = 0, \cdots, B_k(u_1, u_2, \cdots, u_{n-1}) = 0. \]

It follows from Theorem A that for sufficiently large \( n \) it is possible to take \( u_0, u_1, \cdots, u_{n-1} \) in a field obtained from the field of the rational functions of \( a_1, a_2, \cdots, a_n \) by adjunction of a finite number of radicals. The equation \( g(x) \) then has the form
\[ x^n + b_{k+1}x^{n-k} + \cdots + b_n = 0. \]

Its roots then may be considered as algebraic functions of \( n-k \) quantities \( b_{k+1}, b_{k+2}, \cdots, b_n \). Since \( \omega_i \) can be expressed in terms of \( \theta_i \), it follows that the solution of the general equation of \( n \)th degree can be expressed in terms of the coefficients if we use radicals and one algebraic function of \( n-k \) arguments. Here \( k \) was a fixed number and \( n \) was to be taken sufficiently large.

Hilbert's resolvent problem deals with the question of finding the smallest number \( l_n \) for given \( n \) such that the roots of the general equation of degree \( n \) may be expressed in terms of the coefficients by means of algebraic functions of at most \( l_n \) parameters. Our above remark shows that \( l_n \leq n-k \) for fixed \( k \) and sufficiently large \( n \). In other words, we have shown that

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* Since we can make \( b_n = 1 \) through a simple transformation, we could replace the last function by one depending on \( n-k-1 \) arguments.

* This result shows that in Segre's notation an infinite series of theorems \( H_i \) exists. The same is true for the theorems \( B_i \) if in the statement beside the adjunction of square roots and cube roots the adjunction of a finite number of other radicals is admitted. On the other hand, icosahedral irrationalities are superfluous. The existence of these infinite series of theorems \( H_i \) and \( B_i \) had been stated as a conjecture in Segre's paper.
\[ \lim_{n \to \infty} (n - l_n) = \infty. \]

Hilbert's observation that \( l_n \leq n - 5 \), at least for \( n \geq 9 \), and Segre's observation that \( l_n \leq n - 6 \), at least for \( n \geq 157 \), can be supplemented by an infinite number of analogous observations. The method of §2 would allow us to find explicit values \( n_k \) such that \( l_n \leq n - k \) for \( n \geq n_k \). However, the values obtained would probably be far too large.

As an example of a field which satisfies the assumption (*) of Theorem C, we may take any field \( K \) which is closed with regard to forming radicals \( a^{1/m}, a \in K, m = 2, 3, 4, \ldots \). We have here \( \Psi(r) = 2 \) for all \( r \).

In particular, any homogeneous equation \( f(x_1, x_2, \ldots, x_n) = 0 \) of degree \( r \) has a non-trivial solution, provided that \( n \) lies above a certain number depending on \( r \) only.

An example of a somewhat less trivial nature is obtained by considering a \( p \)-adic field \( K \). As is well known the multiplicative group of all \( \alpha^r (\alpha \neq 0, \alpha \in K) \) is of finite index in the group of all \( \alpha (\alpha \neq 0, \alpha \in K) \). From this it follows at once that the assumption (*) of Theorem C is satisfied, and the statement of Theorem C holds for \( K \). In particular, a homogeneous equation \( f(x_1, \ldots, x_n) = 0 \) of degree \( r \) in a \( p \)-adic field has a non-trivial solution \( (x_1, x_2, \ldots, x_n) \), if \( n \) is sufficiently large, say \( n \geq N(r) \).

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10 The somewhat rough method of our proof does not allow us to derive this result. The bound obtained for \( n \) would be much larger.

11 E. Artin has remarked that it follows at once from the existence of normal division algebras of rank \( r^2 \) over \( K \) that \( N(r) > r^2 \).