ON ISOMETRIES OF SQUARE SETS

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1. Introduction. It is not fully known under what conditions the isometry of two square, metric sets, say $E^2$ and $F^2$, implies the isometry of $E$ and $F$. Using the notion of order two self-isometries, this paper gives conditions sufficient to imply $E$ isometric to $F$ when $E^2$ and $F^2$ are finite and are metrized under any one of a fairly extensive class of functions. The basic ideas are first applied to non-square sets to yield a more general theorem which is then applied to the inverse square problem.

2. Definitions. A set is called metric if to every pair of its elements, $a$ and $b$, there corresponds a real, non-negative number, which is independent of the order of $a$ and $b$, zero if and only if $a$ equals $b$, and which satisfies the triangle law.

Two metric sets are isometric (written “$\equiv$”) if there is a one-to-one transformation of one set on the other in which the metric number associated with any pair is the same as that associated with the transformed pair.

A non-identity mapping of a set on itself, which is an isometry, and which leaves each element of the set invariant or else interchanges it with another, is called a self-isometry of order two. Any subset on which the self-isometry is the identity is said to be left pointwise invariant.

Theorem 1. Assume $A \equiv B$ under a mapping $T$, where $A$ and $B$ are finite metric sets. Let $A$ and $B$ have self-isometries of order two under mappings $R$ and $S$ respectively and let $A_1$ and $B_1$ denote respectively the maximum subsets left pointwise invariant. If $A_1$ has no self-isometry of order two, and has at least as many elements as $B_1$, then $A_1 \equiv B_1$ and there

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exists a composition of $R$, $S$, $T$ and $T^{-1}$ which maps $A$ isometrically on $B$ and carries $A_1$ into $B_1$.

**Proof.** Starting with the set $A_1$ the following sequence of sets outlined is obtained by transforming $A_1$ by $T$, the set obtained by $S$, this set by $T^{-1}$, and this set by $R$, and so on repeating cyclically the transformations $T$, $S$, $T^{-1}$, $R$.

<table>
<thead>
<tr>
<th>Column 1</th>
<th>Column 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$T(A_1)$</td>
</tr>
<tr>
<td>$(2, a)$</td>
<td>$T^{-1}ST(A_1)$</td>
</tr>
<tr>
<td>$(2, b)$</td>
<td>$RT^{-1}ST(A_1)$</td>
</tr>
<tr>
<td>$(2n, a)$</td>
<td>$STRT^{-1}ST(A_1)$</td>
</tr>
<tr>
<td>$(2n, b)$</td>
<td>$STRT^{-1}ST(A_1)$</td>
</tr>
<tr>
<td>$(2n + 1, a)$</td>
<td>$(2n + 1 + 1, a)$</td>
</tr>
<tr>
<td>$(2n + 1, b)$</td>
<td>$(2n + 1 + 1, b)$</td>
</tr>
</tbody>
</table>

The notation at the side is such that set $(n, x), x = a$ or $b$, is in $B$ if $n$ is odd and in $A$ if $n$ is even. From the construction and the nature of $R$ and $S$, the following relations are easily verified: $R(2n, a) = (2n, b)$, $R(2n, b) = (2n, a)$, $S(2n + 1, a) = (2n + 1, b)$, $S(2n + 1, b) = (2n + 1, a)$, $T(2n, b) = (2n + 1, a)$, $T^{-1}(2n + 1, b) = (2n + 2, a)$.

*(1)* Assume no set in column 2 is the set $B_1$.

*(2)* Since all sets in both columns are isometric to $A_1$, isometry being transitive, and since $A_1$ has as many elements as $B_1$, (1) implies that no set in column 2 is a subset of $B_1$.

*(3)* For any $n$, $x = a$ or $b$, $S(2n + 1, x) \neq (2n + 1, x)$. Since $S$ is the identity mapping only on $B_1$ and since, from (1) and (2), $(2n + 1, x)$ is not $B_1$ or a subset of it, $S(2n + 1, x) = (2n + 1, x)$ would mean that $(2n + 1, x)$ had a self-isometry of order two. This, together with $A_1 = (2n + 1, x)$, would imply $A_1$ had a self-isometry of order two, contradicting the given conditions.

*(4)* For any $n$, no two sets of column 1 up to and including $(2n, a)$ are identical. The proof is by induction.

*(4.1)* Statement (4) holds for $n = 1$, since $A_1 = (2, a)$ would give $T(A_1) = T(2, a)$ or $(1, a) = (1, b)$, contradicting (3).

*(4.2)* Assume (4) holds for $n = k$.

*(4.3)* Since $R$ is the identity only on $A_1$ and since $(2k, a)$ is not a subset of $A_1$, being isometric to it, and is not equal to $A_1$, from (4.2), then $R(2k, a) = (2k, a)$ would imply that $(2k, a)$ had a self-isometry of
order two, and hence that \(A_1\) did also. Therefore \(R(2k, a) \neq (2k, a)\), that is \((2k, b) \neq (2k, a)\). This, in turn, implies \((2k, b) \neq A_1\).

(4.4) For \(i < k\), \(x = a\) or \(b\), \((2k, b) \neq (2i, x)\). From \((2k, b) = (2i, x)\) would follow \(R(2k, b) = R(2i, x)\), that is \((2k, a) = R(2i, x)\), which for \(i < k\) would contradict (4.2).

(4.5) From (4.2), (4.3), and (4.4) no two sets of column 1 up to and including \((2k, b)\) are identical. This, with the one-to-oneness of \(T\), implies that no two sets of column 2 up to and including \((2k + 1, a)\) are identical.

(4.6) From (3), \((2k + 1, b) \neq (2k + 1, a)\).

(4.7) For \(i < k\), \(x = a\) or \(b\), \((2k + 1, b) \neq (2i + 1, x)\). For, from \((2k + 1, b) = (2i + 1, x)\) would follow \(S(2k + 1, b) = S(2i + 1, x)\), that is \((2k + 1, a) = S(2i + 1, x)\), which for \(i < k\) would contradict (4.5).

(4.8) From (4.6) and (4.7) no two sets of column 2 up to and including \((2k + 1, b)\) are identical. This, with the one-to-oneness of \(T^{-1}\), implies that no two sets of column 1 up to and including \((2k + 1, a)\) are identical, and completes the induction establishing (4).

(5) Since (4) implies the existence of an unlimited number of distinct subsets in the finite set \(A\), it is clearly a contradiction reached through assuming (1). Therefore (1) is false and \(B_1\) must occur in column 2 and be isometric to \(A_1\). The remainder of the theorem follows from the fact that the sequence of sets can be started with \(A\) rather than \(A_1\).

If \(A\) and \(B\) are the same set and \(T\) is replaced by the identity, Theorem 1 reduces to the following result:

**Theorem 2.** Let \(A\) be a finite metric set and let \(A_1\) and \(B_1\) be the maximum subsets left pointwise invariant under two distinct self-isometries, \(R\) and \(S\), of order two. If \(A_1\) has no self-isometry of order two and has at least as many elements as \(B_1\), then \(A_1 = B_1\) and there is a composition of \(R\) and \(S\) which maps \(A\) isometrically on itself and carries \(A_1\) into \(B_1\).

3. **Definitions concerning square sets.** Let \(E\) be a finite metric set with elements \(x_1, x_2, \ldots, x_n\) and metric \(\rho_E\). By \(E^2\) is meant the set of couples obtained from the cartesian product of \(E\) with itself.

In \(E^2\) the subset of couples \((x_i, x_i), i = 1, 2, \ldots, n\), is called the diagonal set.

The reflection mapping, \(R\), of \(E^2\) on itself is defined by \(R(x_i, x_i) = (x_i, x_i)\).

If a metric \(\rho_E^2\) is defined on the elements of \(E^2\) it is called a metric of class \(\alpha\) if, in addition to making \(E^2\) a metric set, it has the following properties:
(1) For any two points of $E^2$, $P_1: (x_i, x_j)$, $P_2: (x_i, x_j)$, $\rho_E^2(P_1, P_2) = f(X_1, X_2)$ where $X_1 = \rho_E(x_i, x_h)$, $X_2 = \rho_E(x_i, x_i)$.

(2) $f(X_1, X_2) = f(X_2, X_1)$.

(3) There exists a constant $M$ associated with $f$, such that whenever $X_1 = X_2$, then $f(X_1, X_2) = MX_1$.

**Theorem 3.** Let $E$ and $F$ be finite metric sets, and let $E^2$ and $F^2$ be metrized under the same class $\alpha$ metric. If either the diagonal set of $E^2$ or that of $F^2$ has no self-isometry of order two, then $E^2 \equiv F^2$ implies $E \equiv F$.

**Proof.** Let $R$ and $S$ denote respectively the reflection mappings of $E^2$ and $F^2$ on themselves. From the definition of reflection and from property 2 of a class $\alpha$ metric, the mappings $R$ and $S$ establish self-isometries of order two in which the diagonal sets alone are left pointwise invariant. The two diagonal sets also have the same number of elements because $E^2 \equiv F^2$. From Theorem 1, then, with $E^2$ and $F^2$ playing the roles of $A$ and $B$, and with the diagonal sets as $A_1$ and $B_1$, it follows that the diagonal set of $E^2$ is isometric to that of $F^2$. This isometry together with property 3 of a class $\alpha$ metric implies $E \equiv F$.

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