ON THE HAUSDORFF DIMENSION OF SOME SETS IN EUCLIDEAN SPACE

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Let E be a closed set in n-dimensional space, x a point not in E. Denote by $S(x)$ the largest sphere of center x which does not contain any point of E in its interior. Put $\phi(x) = E \cap \overline{S}(x)$. ($\overline{A}$ denotes the closure of A.) Denote by M_k the set of points for which $\phi(x)$ contains k or more linearly independent points (that is, points which do not lie in any $(k-2)$-dimensional hyperplane). M_k is defined for $k \leq n+1$. In a previous paper I proved that M_2 has n-dimensional measure 0 and conjectured that M_k has Hausdorff dimension not greater than $n+1-k$. In the present note we shall prove this conjecture. In my previous paper I also proved that M_{n+1} is countable, but the proof there given applied only for the case $n=2$; now we are going to give a general proof.

Let R be any set in n-dimensional space. Let $x \in R$. We define the contingent[1] of R at x (contg$_R$ x) as follows: The contingent will be a subset of the unit sphere. A point z of the unit sphere belongs to contg$_R$ x if and only if there exists a sequence of points $y_1, y_2, \cdots$ in R converging to x so that the direction of the vector connecting x with y_j tends to the direction of the vector connecting the center of the unit sphere with z. First we state the following lemma.

LEMMA. *Let there be given a set R in n-dimensional space. Assume that for every x, contg$_R$ x does not contain any point of the intersection of the unit sphere with a k-dimensional hyperplane going through its center (the hyperplane can depend on x). Then R is contained in the sum of countably many surfaces of finite $(n-k)$-dimensional measure.*

This lemma is well known.[2]

THEOREM 1. *Let $k < n+1$. Then M_k is contained in the sum of countably many surfaces of finite $(n+1-k)$-dimensional measure. If $k = n+1$, then M_k is countable.*[3]

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[1] G. Bouligand, *Introduction à la géometrié infinitésimale directe*. Also Saks, *Theory of the integral.*

[2] Saks, ibid. pp. 264–266 and pp. 304–307. Also Roger, C. R. Acad. Sci. Paris vol. 201 (1935) pp. 871–873.

[3] For $n=2$ this theorem is proved by C. Pauc, Revue Scientifique, August, 1939.

Remark. This clearly means that the Hausdorff dimension of M_k $(k \leq n+1)$ is not greater than $n+1-k$.

Let us first consider the case $k = n+1$. Assume that $x \in M_{n+1}$. Let $z_i \in \phi(x)$, $i = 1, 2, \cdots, n+1$, and assume that the z's are linearly independent. Denote by $f(x)$ the maximum value of the volume of the simplices determined by the z's (since $\phi(x)$ is closed the maximum is attained). Define now $N_{n+1}^{(c)} = N$ to consist of all the points $x \in M_{n+1}$ for which $f(x) \geq c$. It clearly will be sufficient to show that N is countable (for every c). In fact we shall show that N is isolated (in other words no $x \in N$ is a limit point of $N - x$), that is, we shall prove that for every $x \in N$ contg$_N$ x is empty. If this would not hold then N would contain an infinite sequence of points y_j covering to x so that the direction of the line connecting x with y_j would converge to a fixed direction. Let Z_j be a point of $\phi(x)$ which is closest to y_j, and let A_j be the (unique) hyperplane through Z_j perpendicular to the segment xy_j. It is easy to see that as $j \to \infty$, A_j converges to a limiting hyperplane A. Moreover it is easily seen that the set $\phi(y_j)$ is ultimately contained in any preassigned neighborhood of A. Thus for large enough j, the volume $f(y_j)$ must be less than c, an evident contradiction; this completes our proof.

Next we prove our theorem in the general case. Let $k \leq n$ and define M_k' to be the set of all points x for which the maximum number of linearly independent points in $\phi(x)$ is exactly k. It will clearly be sufficient to show that M_k' is contained in the sum of countably many surfaces of finite $(n+1-k)$-dimensional measure. Let $x \in M_k'$, and let $f(x)$ be the maximum volume of the k-dimensional simplices formed from the points z_i, $i \leq k+1$, where $z_i \in \phi(x)$. $x \in M_k'^{(c)} = N'$ if $f(x) \geq c$. Let $x \in N'$, and z_i, $i \leq k+1$, be the points which determine a simplex of maximal volume. Then a simple geometrical argument (similar to the previous one) shows that contg$_{N'}$ x consists only of the directions through x which are perpendicular to the hyperplane determined by the z_i's, $i \leq k+1$. Thus our theorem follows from the lemma.

Let E be a closed set, $x \notin E$. Denote by $g(x)$ the distance of x from E. It has been proved[4] that $g(x)$ has a derivative $-\cos \alpha$ in every direction (x, y), where α is the smallest angle formed by the direction (x, y) with the direction (x, z), z in $\phi(x)$. Clearly if $x \in E$ the derivative of $g(x)$ can be 0. We shall show that the derivative of $g(x)$ is 0 for almost all points of E.

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[4] Mises, C. R. Acad. Sci. Paris vol. 205 (1937) pp. 1353–1355. See also Golab, ibid. vol. 206 (1938) pp. 406–408 and Bouligand, ibid. vol. 206 (1938) pp. 552–554.

Let $x \in E$. Denote by $S(x, \epsilon)$ the sphere of center x and radius ϵ. Denote by $G(x, \epsilon)$ the greatest distance of the points of $\overline{S}(x, \epsilon)$ from E. We are going to prove the following theorem.

THEOREM 2. *For almost all points of E (that is, for all points of E except a set of n-dimensional measure* 0)

$$\lim G(x, \epsilon)/\epsilon = 0.$$

It is well known that almost all points of E are points of Lebesgue density 1. Let x be such a point, and suppose that

$$\lim G(x, \epsilon)/\epsilon \neq 0.$$

This means that there exists an infinite sequence ϵ_i and points z_i, $z_i \in \overline{S}(x, \epsilon_i)$, $\epsilon_i \to 0$, such that the distance of z_i from E is greater than $c\epsilon_i$, where $c > 0$. But this clearly means that x can not have Lebesgue density 1. This contradiction establishes our theorem.

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ON MONOTONE RETRACTABILITY INTO SIMPLE ARCS

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In recent work on the area of surfaces Radó [1][1] has had occasion to use the following properties as applied to locally connected continua A:

(π) *Every simple arc in A is a monotone retract of A*;

(II) *Every monotone image of A has property* (π).

Radó has noted that (II) implies (π) and that the sphere and 2-cell each have (II). In this paper it will be shown that (1) for locally connected continua in general, property (II) is equivalent to unicoherence, (2) for plane locally connected continua, property (π) is equivalent to unicoherence, and (3) every closed 2-dimensional connected manifold has property (π).

To clarify our meaning, we recall that a continuum is compact, connected and metric. A continuous mapping $f(A) = B$ on a continuum A is *monotone* provided $f^{-1}(y)$ is a continuum for $y \in B$. If

[1] Numbers in brackets refer to the Bibliography at the end of the paper.