A NOTE ON WEAK DIFFERENTIABILITY OF PETTIS INTEGRALS

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Pettis\(^1\) raised the question whether or not separability of the range space implies almost everywhere weak differentiability of Pettis integrals. Phillips\(^2\) has given an example which answers this question in the negative. His construction is based on a sequence of orthogonal vectors in Hilbert space. We present here a different example of the same type of function. Our basic construction is that of a function defined to the space \(C\). Using that function as a basis, we are able to give a specific construction of such a function defined to each member of a large class of Banach spaces.

1. Metric density properties of a non-dense perfect set. Let \(B \subseteq [0, 1]\) be a non-dense perfect set of measure one-half, and let \(\overline{B}\) be its complement. \(\overline{B}\) may be constructed by taking the sum of a set of open intervals classified as follows:

1 interval of length 1/4,
2 intervals each of length 1/16,
4 intervals each of length 1/64,
\[\ldots\]
2\(^n-1\) intervals each of length 1/2\(^{2n}\),
\[\ldots\]

We shall refer to the intervals of length 1/2\(^{2n}\) as intervals of \(\overline{B}\) of order \(n\). We shall assume that each interval of \(\overline{B}\) of order \(n\) is the center portion of the space either between two intervals of \(\overline{B}\) of lower order or between one such interval of \(\overline{B}\) and an end point of the unit interval. These spaces we shall refer to as gaps of order \(n\), and we shall denote such a gap by the symbol \(G_n\). If \(\overline{B}\) is constructed as noted above, then for each \(n\), any two sets each of the form \(G_n\). \(\overline{B}\) are congruent; hence we shall use \(G_n\) to denote a gap of order \(n\), and we shall not find it necessary to specify which one.

The following three lemmas are now obvious.

1.1. Lemma. \(|\overline{B} \cdot G_n| = 1/2^{2n-1}|.\)

1.2. Lemma. \(|G_n| = 1/2^n + 1/2^{2n-1}|.

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\(^1\) See [3, p. 303]. Numbers in brackets refer to the references cited at the end of the paper.

\(^2\) See [4, p. 144].
1.3. Definition. If $I$ is any subinterval of $[0, 1]$, we define

$$\rho(I) = \frac{|B \cdot I|}{|I|}.$$ 

1.4. Lemma. $\rho(G_n) = 1/(1 + 2^{n-1})$.

The following important lemma demonstrates a lower bound on the density function $\rho$. This lower bound may tend to zero as $|I|$ tends to zero, but it is independent of the location of $I$.

1.5. Lemma. If $|I| \geq |G_n| = 1/2^n + 1/2^{2n-1}$,

then

$$\rho(I) > \frac{1/2^{2n-2}}{1/2^n + 1/2^{2n-1}} = \frac{2}{1 + 2^{n-1}} = 2\rho(G_n).$$

If $I$ is moved on beyond this interval of $B$ the above argument applies again by considering the movement in the reverse direction. Comparison with Lemma 1.4 now shows that Lemma 1.5 is established in case $|J| = |G_n|$.

Consider now the effect of increasing $|I|$. If $|I| = |G_n|$ and $I$ contains no interval of $B$ of order less than or equal to $n-1$, then one end point of $I$ (let us assume it is the right-hand one) lies in the closure of such an interval of $B$. Thus any small extension of $I$ to the right will (until the interval of $B$ is covered) add to $I$ only points of $B$, thus obviously increasing $\rho(I)$. If $I = G_n$ at the start, a small extension to the left will have the same effect. Otherwise an extension to the left may be regarded as a translation to the left and a subsequent extension to the right, and these cases have already been discussed. In case $I$ contains an interval of $B$ of order not greater than $n-1$, we have $|B \cdot I| > 1/2^{2n-2}$; thus if $|I| \leq |G_{n-1}| = 1/2^{n-1} + 1/2^{2n-2}$,

$$\rho(I) > \frac{1/2^{2n-2}}{1/2^{n-1} + 1/2^{2n-2}} = \frac{1}{2 + 2^{n-1}}.$$ 

If $|I| > |G_{n-1}|$, the above argument may be repeated with $n-1$ sub-
stituted for \( n \), thereby obtaining an even larger lower bound for \( \rho(I) \). This proves Lemma 1.5 for all cases.

1.6. Lemma. For all \( I \subseteq [0, 1] \),

\[
\rho(I) > | \mathcal{B} \cdot I |^{1/2}/2^{1/2}.
\]

We shall prove Lemma 1.6 by showing that, for each \( n \), the required inequality holds for

\[
1/2^{2n+1} < | \mathcal{B} \cdot I | \leq 1/2^{2n-1}.
\]

This will cover all possibilities. For \( | \mathcal{B} \cdot I | \) in this range, we consider first the case \( | I | < | G_n | = 1/2^n + 1/2^{2n-1} \). In this case

\[
\frac{1}{\rho(I)} = \frac{| I |}{| \mathcal{B} \cdot I |} < \frac{1/2^n + 1/2^{2n-1}}{1/2^{2n+1}} = 4 + 2^{n+1}
\]

\[
= 4 + 2^{3/2}[1/2^{2n-1}]^{-1/2} \leq 4 + 2^{3/2} | \mathcal{B} \cdot I |^{-1/2}.
\]

Considering now the case \( | I | \geq | G_n | \) (and assuming \( | \mathcal{B} \cdot I | \) still in the same range) we have, using Lemma 1.5 and the above inequalities,

\[
1/\rho(I) < 2 + 2^{n-1} \leq 4 + 2^{3/2} | \mathcal{B} \cdot I |^{-1/2}.
\]

Now for all \( I \subseteq [0, 1] \), \( | \mathcal{B} \cdot I | \leq 1/2 \); hence \( | \mathcal{B} \cdot I |^{-1/2} \geq 2^{1/2} \); hence \( 4 \geq 2^{3/2} | \mathcal{B} \cdot I |^{-1/2} \). Combining this with the above results, we have

\[
1/\rho(I) < 4 + 2^{3/2} | \mathcal{B} \cdot I |^{-1/2} \leq 2^{3/2} | \mathcal{B} \cdot I |^{-1/2},
\]

whence

\[
\rho(I) > | \mathcal{B} \cdot I |^{1/2}/2^{1/2}.
\]

2. An approximately continuous function whose integral is non-differentiable. For each \( t \in \mathcal{B} \) we define the function \( f_t(x) \) for \( x \in [0, 1] \) as follows

\[
f_t(x) = \begin{cases} 
0 & \text{for } x \leq t \text{ or } x \in \mathcal{B}, \\
| \mathcal{B} \cdot [t, x] |^{-1/4} & \text{for } x > t \text{ and } x \in \mathcal{B}.
\end{cases}
\]

2.1. Theorem. For each \( t \in \mathcal{B} \), \( f_t(x) \) is an integrable function of \( x \), and for \( x_2 \geq x_1 \geq t \),

\[
\int_{x_1}^{x_2} f_t(x) \, dx = 4( | \mathcal{B} \cdot [t, x_2] |^{1/4} - | \mathcal{B} \cdot [t, x_1] |^{1/4}).
\]

Let \( z = | \mathcal{B} \cdot [t, x] | \). Since the intervals of \( \mathcal{B} \) are dense in \( [0, 1] \), this defines \( z \) as a strictly monotone function of \( x \); hence \( x \) is a single-valued function of \( z \), and we may write \( f_t(x(z)) \). Now for \( x > t \) and \( x \in \mathcal{B} \), \( dx = dz \); thus the function \( s(x) \) is measure preserving over \( \mathcal{B} \).

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· [t, x] and hence maps $B · [t, x]$ into a set of measure zero. Therefore, for almost all $z$, $f_i[x(z)] = z^{-3/4}$; and $dx = ds$ except where $f_i(x) = 0$; so

$$\int_{z_1}^{z_2} f_i(x)dx = \int_{z_1}^{z_2} z^{-3/4} dz = 4(z_2^{1/4} - z_1^{1/4}).$$

2.2. Theorem. For $t \in B$, the function

$$F_i(x) = \int_t^x f_i(u)du$$

is not differentiable with respect to $x$ at $x = t$.

Again letting $z = |B · [t, x]|$, and using Theorem 2.1 and Lemma 1.6, we have

$$\frac{F_i(x)}{x - t} = \frac{4z^{1/4}}{x - t} = 4z^{-3/4} \left( \frac{z}{x - t} \right) = 4z^{-3/4} \rho([t, x])$$

$$> 4z^{-3/4}(z^{1/2}/2^{5/2}) = (4z)^{-1/4}.$$ 

Thus

$$\limsup_{z \to t} \frac{F_i(x)}{x - t} \geq \lim_{z \to 0} (4z)^{-1/4} = \infty.$$

In the next section we shall make further use of the functions $f_i(x)$ and their properties as shown in Theorems 2.1 and 2.2. We might note here, however, that $f_i(x)$ is approximately continuous at $t$ provided $B$ has metric density zero at $t$. This is true for almost all $t$ in $B$; hence for such $t$, $f_i(x)$ furnishes a specific example of an approximately continuous function whose integral is not differentiable.

3. A Pettis integral in the space $C$ which is not almost everywhere weakly differentiable. We shall here define a function $\phi(x)$ from $[0, 1]$ to the space $C$. Our notation will be as follows: For each $x \in [0, 1]$, $\phi(x)$ stands for a continuous function on $[0, 1]$; we denote this continuous function by $\phi_x(t)$. We shall define the functions $\phi_x(t)$ by defining a function $\phi(x, t)$ over the unit square and setting $\phi_x(t) = \phi(x, t)$. We first define $\phi(x, t)$ over a portion of the unit square as follows:

$$\phi(x, t) = \begin{cases} 0 & \text{for } x \in B, \\ f_i(x) & \text{for } t \in B. \end{cases}$$

Since $f_i(x) = 0$ for $x \in B$, these statements are consistent.

3.1. Lemma. For a fixed $x$, $\phi(x, t)$ is continuous in $t$ over $B$.

This statement follows immediately from the fact that if one end
point of \( I \) is fixed, \(|B \cdot I|^{-1/4} \) is a continuous function of the other end point over any set such that \(|I|\) is bounded away from zero. This latter restriction causes no difficulties here. If \( x \in B, \phi(x, t) = 0; \) if \( x \in \overline{B}, \) dist \((x, B) > 0\).

We now continue the definition of \( \phi(x, t) \). For each \( x \in \overline{B}, \) let \( \phi(x, t) \) be continued linearly over each interval of the set \( t \in \overline{B}. \) This completes the definition of \( \phi(x, t) \) over the entire unit square, and it is clear that for each \( x, \phi(x, t) \) is continuous in \( t \) over \([0, 1]\).

3.2. **Theorem.** \( \phi(x) \) is integrable in the sense of Pettis. For each measurable set \( E \subseteq [0, 1], \) its integral over \( E \) is the function

\[
\Phi_E(t) = \int_E \phi(u, t) \, du.
\]

We show this by considering the functions \( \phi^{(n)}(x) \) whose values are the continuous functions \( \phi^{(n)}(x) = \phi^{(n)}(x, t) \) where

\[
\phi^{(n)}(x, t) = \begin{cases} \phi(x, t) & \text{for } \phi(x, t) \leq n, \\ n & \text{for } \phi(x, t) > n. \end{cases}
\]

It is easily seen that for each \( t \in [0, 1], \) each \( \phi^{(n)}(x, t) \) is bounded and continuous in \( x \) over \( \overline{B}. \) Thus\(^5\) each \( \phi^{(n)}(x) \) is weakly continuous over \( \overline{B}. \) Since \( \phi^{(n)}(x) = 0 \) for \( x \in B, \) it is clear that each \( \phi^{(n)}(x) \) is weakly measurable. Since \( C \) is a separable space, it follows\(^4\) that each \( \phi^{(n)}(x) \) is measurable. Now each \( \phi^{(n)}(x) \) is bounded, hence Bochner integrable, hence Pettis integrable, therefore integrable with respect to each of the linear functionals \( \gamma_t[\phi^{(n)}(x)] = \phi^{(n)}_x(t); \) thus

\[
\Phi^{(n)}(E) = \int_E \phi^{(n)}(u) \, du
\]

is the continuous\(^6\) function

\[
\Phi^{(n)}_{E}(t) = \int_E \phi^{(n)}(u, t) \, du.
\]

Clearly for each \( x \) and each \( t,\)

\[
\lim_{n \to \infty} \phi^{(n)}_x(t) = \phi_x(t);
\]

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\(^*\) See [1, p. 224, Theorem 8].

\(^4\) See [3, Theorem 1.1].

\(^5\) An independent proof of continuity of \( \Phi^{(n)}_{E}(t) \) is unnecessary. Since \( \phi^{(n)}(x) \) is Pettis integrable, it is integrable to an element of \( C; \) and the set \( \{ \gamma_t \} \) of linear functionals defines this element uniquely.
Furthermore, this approximation is monotone in $n$. Thus $||\phi^{(n)}(x) - \phi(x)||$ is bounded in $n$ for each $x$; hence $\phi^{(n)}(x) \rightarrow \phi(x)$ weakly for each $x$. It now follows that $\phi(x)$ is Pettis integrable provided the sequence $\{\phi^{(n)}(E)\}$ converges with respect to the norm in $C$; that is, provided $\{\Phi^{(n)}(t)\}$ converges uniformly in $t$. We shall complete the proof of Theorem 3.2 by showing that for each measurable $E \subset [0, 1]$

$$\Phi^{(n)}(t) = \int_E \phi(u, t) du$$

exists for each $t$ and that this function is the uniform limit of the sequence $\{\Phi^{(n)}(t)\}$.

To show that $\Phi^{(t)}(t)$ exists is trivial. For $t \in B$, this follows from Theorem 2.1. For each $x$, $\phi(x, t)$ is extended linearly over each interval of $t \in \overline{B}$; hence for $t \in \overline{B}$, $\phi(x, t) \leq \phi(x, t_1) + \phi(x, t_2)$ where $t_1$ and $t_2$ are each in $B$. This completes the proof of integrability.

Now with each $t \in [0, 1]$ we associate two numbers $t_1$ and $t_2$ as follows: $t_1$ is the greatest number such that $t_1 \in B$ and $t \leq t_1$; $t_2$ is the smallest number such that $t_2 \in B$ and $t \geq t_2$. Geometrically this means that if $t \in B$, $t_1 = t = t_2$, while if $t \in \overline{B}$, $t_1$ and $t_2$ are the left and right points respectively of the interval of $\overline{B}$ in which $t$ is located.

Now for $t_1 \leq x < t_2$, $\phi(x, t) \leq \phi(x, t_1)$ while for $x \geq t_2$, $\phi(x, t) \leq \phi(x, t_2)$. Thus for any given $t \in [0, 1]$, it is possible to have $\phi(x, t) > n$ only for those values of $x$ for which either

$$t_1 \leq x < t_2 \text{ and } |\overline{B} : [t_1, x]| < n^{-4/3}$$

or

$$x \geq t_2 \text{ and } |\overline{B} : [t_2, x]| < n^{-4/3}.$$ 

Outside these two intervals $\phi(x, t) - \phi^{(n)}(x, t) = 0$; hence if we denote these intervals by $I_1$ and $I_2$, we have

$$\int_{I_1} [\phi(x, t) - \phi^{(n)}(x, t)] dx \leq \int_{I_1} \phi(x, t) dx + \int_{I_2} \phi(x, t) dx$$

$$\leq \int_{I_1} \phi(x, t_1) dx + \int_{I_2} \phi(x, t_2) dx$$

$$< 2 \int_0^{n^{-4/3}} x^{-3/4} dx = 8n^{-1/3}.$$ 

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6 See [1, p. 224, Theorem 8].
7 See [3, Theorem 4.1].
Thus, clearly, \( \lim_{n \to \infty} \Phi_E^{(n)}(t) = \Phi_E(t) \) uniformly in \( t \). This completes the proof of Theorem 3.2.

3.3. Theorem. If \( x_0 \in B \), \( \Phi(x) = \int_0^x \phi(u)du \) is not weakly differentiable at \( x_0 \).

By Theorem 2.2, it fails to be differentiable at \( x_0 \) with respect to the linear functional \( \gamma_{x_0}[\Phi(x)] = \Phi(x_0) \).

4. Extension to other spaces of continuous functions. The function \( \phi(x) \) of §3 may be used as the basis for the construction of a large set of examples as follows:

4.1. Theorem. If \( \Omega \) is a compact metric space containing non-denumerably many points and if \( C(\Omega) \) is the Banach space of all continuous functionals on \( \Omega \), then there is a function \( \psi(x) \) from the unit interval to \( C(\Omega) \) such that \( \psi(x) \) is Pettis integrable but \( \Psi(E) = \int_E \psi(x)dx \) fails to be weakly differentiable on a set of positive measure.

Since \( \Omega \) is non-denumerable, it contains a perfect set. This perfect set is a complete metric space which is dense in itself and hence contains a homeomorph \( \Pi \) of the Cantor set \( B \).

Let \( B = h(\Pi) \) be the homeomorphic mapping of \( \Pi \) into \( B \). Then \( h(\omega) \) is a continuous function defined over \( \Pi \), assuming values between 0 and 1, and assuming for some \( \omega \in \Pi \) each value in the set \( B \).

Let \( H(\omega) \) be a continuous extension\(^8\) of \( h(\omega) \) over the whole of \( \Omega \) with \( 0 \leq H(\omega) \leq 1 \).

Now for each \( t \in [0, 1] \) we define
\[
K(t) = \{ \omega : H(\omega) = t \}.
\]
It should be noted that although for some \( t \), \( K(t) \) may be vacuous, for each \( t \in B \), \( K(t) \) contains at least one point.

Referring back to the functions \( \phi_x(t) \) of §3, we now define
\[
\psi_x(\omega) = \phi_x(t) \quad \text{for} \quad \omega \in K(t).
\]
It follows from the continuity in \( t \) of each function \( \phi_x(t) \) and from the continuity of \( H(\omega) \) that for each \( x \in [0, 1] \), \( \psi_x(\omega) \) is continuous over \( \Omega \). For each \( x \in [0, 1] \) we now let \( \psi(x) \) be the element \( \psi_x(\omega) \) of \( C(\Omega) \).

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\(^8\) See [2, p. 228]. The author is indebted to the referee for the suggestion that non-denumerability of \( \Omega \) is sufficient to insure the existence of \( \Pi \).

\(^9\) See [2, p. 211]. In connection with our remark in the introduction that we have a specific construction applicable to the more general spaces, it should be noted that this extension theorem is not merely an existence proof. A definite formula for the extension is given.
That $\psi(x)$ has the required properties may be seen as follows: To show integrability, we note that for each $\omega \in \Omega$, $\psi_\omega(\omega)$ is identical (as a function of $x$) with $\phi_x(t)$ for some $t \in [0, 1]$. Then noting that Banach's criterion for weak convergence\(^{10}\) applies to the space $C(\Omega)$, we apply the proof of Theorem 3.2. To show non-differentiability, we note that for each $t \in B$ there is an $\omega \in \Omega$ such that $\psi_\omega(\omega) = \phi_x(t)$ for all $x \in [0, 1]$. The proof of Theorem 3.3 then applies.

References


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\(^{10}\) This is used in the proof of Theorem 3.2. See footnotes 3 and 6.