NOTE ON A NOTE OF H. F. TUAN

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The following theorem is proved.

THEOREM. If \( Z \) is a nilpotent matrix with elements in a field \( K \), then the replicas of \( Z \) are those and only those matrices which are of the form \( f(Z) \), where \( f(x) \) is an additive polynomial in \( K[x] \).

The concept of a replica was introduced by Chevalley, who proved this theorem when \( K \) is of characteristic zero. The theorem was proved in general by H. F. Tuan by elementary methods. The object of this note is to give a simplification of Tuan's proof; in particular, computations involving the specific form of \( Z \) are avoided.

If \( h(x) \) is additive, then according as \( K \) is of characteristic 0 or \( p \), \( h(x) \) will have one of the two forms

\[
(1) \quad tx, \quad \sum_{j=1}^{m} t_j x^{pj} \quad (t, t_j \in K).
\]

For if \( h(x) \) had any other terms, then \( h(x) + h(y) = h(x+y) \) would contain product terms \( x^\alpha y^\beta, \alpha > 0, \beta > 0 \). Conversely, polynomials of the form (1) are clearly additive. If \( h(x) = \sum_{k=0}^{s} c_k x^k (c_k \in K) \), then we define

\[
h^{(i)}(x) = \sum_{k=i}^{s} C_{k-i} c_k x^{k-i},
\]

where the \( C_{k,i} \) are binomial coefficients. Evidently

\[
h^{(i)}(x) = i! h^{(i)}(x), \quad h(x + y) = \sum_{i=0}^{s} h^{(i)}(x) y^i.
\]

It follows from this that \( h(x) \) is additive if and only if \( c_0 = 0 \) and \( h^{(i)}(x) = c_i \) for \( i > 0 \).

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1 A polynomial \( f(x) \) is additive if \( f(x+y) = f(x) + f(y) \). The statement of the theorem in terms of the additivity of \( f(x) \) rather than in terms of the explicit form (1), as well as the use of the derived polynomials \( f^{(i)}(x) \) to replace explicit computation with binomial coefficients, was suggested by Professor Jacobson.


If \( f(x) \) is additive then it is easily seen that \( f(Z)_{r,s} = f(Z_{r,s}) \); hence \( (Z) \) is a replica of \( Z \) if \( f(x) \) is additive. The converse follows from:

**Lemma 1.** Let \( Z \) be a nilpotent matrix over a field \( K \) and let \( Z' \) be a matrix such that \( Z' = f(Z), Z'_{r,s} = g(Z_{r,s}) \), where \( f(x) \) and \( g(x) \) are polynomials without constant terms. Then \( f \) may be assumed to be additive.

This lemma does indeed imply the theorem since Chevalley\(^2\) has proved (p. 529) that a replica \( Z' \) of \( Z \) satisfies the hypothesis of the lemma.

We now recall some definitions. If \( A \) and \( B \) are \( n \times n \) matrices over \( K \), then \( A \times B \) is the \( n^2 \times n^2 \) matrix formed by the \( n \times n \) array of matrices \( a_{ij}B \), where \( A = (a_{ij}) \). The following statements are evident:

\[
(A \times B)(C \times D) = AC \times BD,
\]
\[
(A + A_1) \times B = A \times B + A_1 \times B,
\]
\[
cA \times B = c(A \times B) \quad \text{if} \quad c \in K,
\]

(2) \( A \times B = 0 \) implies \( A = 0 \) or \( B = 0 \).

Finally, \( Z_{0,2} \) is defined as \( Z \times E + E \times Z \); \( E \) is the \( n \times n \) unit matrix, where \( n \) is the dimension of \( Z \).

Since \( Z \) is nilpotent and since it may be assumed that \( Z \neq 0 \) (for Lemma 1 is trivial if \( Z = 0 \)), there is an integer \( m \) such that

\[
(3) \quad Z^m \neq 0, \quad Z^{m+1} = 0, \quad 1 \leq m \leq n - 1.
\]

**Lemma 2.** If \( A_0, A_1, \ldots, A_m \) are \( n \times n \) matrices such that

\[
A_0 \times E + A_1 \times Z + \cdots + A_m \times Z^m = 0,
\]

then \( A_0 = A_1 = \cdots = A_m = 0 \).

**Proof.** Multiplying by \( E \times Z^n \), we obtain \( A_0 \times Z^n = 0, A_0 = 0 \) by (2) and (3). Multiplying successively by \( E \times Z^{n-1}, E \times Z^{n-2}, \ldots \), we obtain \( A_1 = A_2 = \cdots = 0 \).

**Proof of Lemma 1.** In view of (3) the polynomial \( f(x) \) may be assumed to be of degree at most \( m \). We show that it must then be additive. Now

\[
(Z_{0,2})^k = (Z \times E + E \times Z)^k = \sum_{i=0}^{k} C_{k,i} Z^{k-i} \times Z^i.
\]

It follows that \( (Z_{0,2})^{2m+1} = 0 \), so that \( g(x) \) may be assumed of degree at most \( 2m \). We observe incidentally that Lemma 2 implies that

\(^4\) Loc. cit., Theorems (B) and (C).
(Z_{0,2})^m \neq 0; \text{ if } K \text{ is of characteristic 0, then also } (Z_{0,2})^{2m} \neq 0, \text{ for }
(Z_{0,2})^{2m} = C_{2m,m}Z^m \times Z^m.\n
Now we have

\begin{equation}
Z'_{0,2} = g(Z_{0,2}) = g(Z \times E + E \times Z) = \sum_{i=0}^{2m} g^{[i]}(Z \times E)(E \times Z)^i
\end{equation}

\begin{equation}
= g(Z) \times E + g^{[1]}(Z) \times Z + g^{[2]}(Z) \times Z^2 + \cdots + g^{[m]}(Z) \times Z^m.
\end{equation}

On the other hand, placing \( f(x) = \sum_{i=1}^{m} a_i x^i \), we have

\begin{equation}
Z'_{0,2} = Z' \times E + E \times Z' = f(Z) \times E + E \times f(Z)
\end{equation}

\begin{equation}
= f(Z) \times E + a_1 E \times Z + a_2 E \times Z^2 + \cdots + a_m E \times Z^m.
\end{equation}

A comparison of (4) and (5) gives, by Lemma 2,

\begin{align*}
g(Z) &= f(Z), \\
g^{[i]}(Z) &= a_i E, \quad i = 1, \ldots, m; \\
g(x) &= f(x)(x^{m+1}), \quad g^{[i]}(x) = a_i(x^{m+1}), \quad i = 1, \ldots, m.
\end{align*}

From the first congruence, \( g^{[i]}(x) \equiv f^{[i]}(x)(x^{m+1-i}) \), and from the second, \( f^{[i]}(x) \equiv a_i(x^{m+1-i}) \). Since \( f^{[i]}(x) \) is of degree at most \( m - i \), \( f^{[i]}(x) = a_i \). By a previous remark it follows that \( f(x) \) is additive, and this completes the proof.

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