

## SOME PROPERTIES OF PARTIAL SUMS OF THE HARMONIC SERIES

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It has been proved that  $\sum_{k=m}^n k^{-1}$  cannot be an integer<sup>1</sup> for any pair of positive integers  $m$  and  $n$ . More generally,  $\sum_{k=0}^n (m+kd)^{-1}$  cannot be an integer.<sup>2</sup> We prove two theorems of a similar nature.

**THEOREM 1.** *There is only a finite number of integers  $n$  for which one or more of the elementary symmetric functions of  $1, 1/2, 1/3, \dots, 1/n$  is an integer.*

**PROOF.** Let  $\sum_{k,n}$  denote the  $k$ th symmetric function of  $1, 1/2, 1/3, \dots, 1/n$ . Since each term of  $\sum_{k,n}$  is contained  $k!$  times in the expansion of  $(1+1/2+\dots+1/n)^k$ , we have, for  $k > 3 \log n$  and  $n$  sufficiently large,

$$\sum_{k,n} < \frac{(1+1/2+\dots+1/n)^k}{k!} < \frac{(1+\log n)^k}{k!} < 1,$$

where the second inequality arises from the usual comparison of  $\log n$  with the harmonic series, and the third inequality is implied by the hypothesis  $k > 3 \log n$ .

Henceforth we take  $k < 3 \log n$ . By a theorem of A. E. Ingham<sup>3</sup> there is a prime between  $x$  and  $x+x^{5/8}$ . This implies that there is a prime  $p$  between  $1+n/(k+1)$  and  $n/k$  for  $k < 3 \log n$  and  $n$  sufficiently large. Hence  $\sum_{k,n}$  contains the term

$$\frac{1}{p} \cdot \frac{1}{2p} \cdots \frac{1}{kp} = \frac{1}{k!p^k}.$$

Now  $(k!, p) = 1$  since  $k < n/(k+1)$ , and hence no other term in  $\sum_{k,n}$  has a denominator divisible by  $p^k$ . So if  $\sum_{k,n} = a/b$ , we know that  $p^k | b$  and  $p \nmid a$ , which proves the theorem.

By a similar but more complicated argument we can prove the same

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<sup>1</sup> Cf. Pólya-Szegő, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, Berlin, 1925, chap. 8, p. 159, problem 250.

<sup>2</sup> Cf. T. Nagell, *Eine Eigenschaft gewissen Summen*, Skrifter Oslo, no. 13 (1923) pp. 10-15.

<sup>3</sup> *On the difference between consecutive primes*, Quart. J. Math. Oxford Ser. vol. 8 (1937) p. 256. This result is actually stronger than necessary for our use here. The classical estimates will suffice.

result for the elementary symmetric functions of  $1/m, 1/(m+1), \dots, 1/n$ , and of  $1/m, 1/(m+d), 1/(m+2d), \dots, 1/(m+nd)$ .

It should be noted that  $\sum_{2,3}$  is an integer; we know of no other integral case. Theorem 1 can be proved without the use of the prime number theorem, and this proof could be used to determine the bound on  $n$ , above which the result of the theorem holds. For smaller values of  $n$ ,  $\sum_{k,n}$  could be checked, but the proof is complicated and the limits would be large.

**THEOREM 2.** *No two partial sums of the harmonic series can be equal; that is, it is not possible that*

$$(1) \quad \begin{aligned} 1/m + 1/(m + 1) + \dots + 1/n \\ = 1/x + 1/(x + 1) + \dots + 1/y. \end{aligned}$$

**PROOF.** We assume that  $n < x$ . Clearly if (1) has a solution, then any prime divisor of one of the denominators must divide another. Hence by Bertrand's postulate we can be certain that  $y < 2x - 1$ , since otherwise a prime  $p > n$  would be one of the denominators on the right side of (1).

**LEMMA.** *Any solution of (1) must satisfy  $y < x + x^{1/2} - 1$ .*

To prove this we use a theorem of Sylvester and Schur<sup>4</sup> which states that if  $n > k$ , then in the set  $n, n+1, \dots, n+k-1$  there is an integer containing a prime divisor greater than  $k$ . In our case  $x > y - x + 1$ , so that there is a prime  $p > y - x + 1$  which divides one and only one (say  $ap$ ) of the integers  $x, x+1, x+2, \dots, y$ . Also  $p$  must divide one (say  $bp$ ) of the set  $m, m+1, m+2, \dots, n$ , and certainly not more than one, since  $n - m < y - x$ . Then  $1/ap$  and  $1/bp$  are the only terms in equation (1) whose denominators are divisible by  $p$ , and since

$$1/bp - 1/ap = (a - b)/abp,$$

we conclude that  $p$  must divide  $a - b$ , whence  $a - b \geq p$  and  $a \geq p + 1$ . This implies that

$$y \geq ap \geq p^2 + p > (y - x + 1)^2 + y - x + 1$$

or

$$x - 1 > (y - x + 1)^2,$$

which proves the lemma.

Next we obtain estimates for the expressions in (1). First we note that

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<sup>4</sup> Cf. Paul Erdős, J. London Math. Soc. vol. 9 (1934) p. 282.

$$\begin{aligned} \log \frac{2k + 1}{2k - 1} &= \log \left( 1 + \frac{1}{2k} \right) - \log \left( 1 - \frac{1}{2k} \right) \\ &= \frac{1}{k} + \sum_{j=1}^{\infty} \frac{2}{(2j + 1)(2k)^{2j+1}}. \end{aligned}$$

Solving for  $1/k$ , and summing the result for  $k = m, m + 1, \dots, n$ , we obtain

$$\begin{aligned} (2) \quad \frac{1}{m} + \frac{1}{m + 1} + \dots + \frac{1}{n} \\ &= \log \frac{2n + 1}{2m - 1} - \sum_{k=m}^n \sum_{j=1}^{\infty} \frac{2}{(2j + 1)(2k)^{2j+1}}, \end{aligned}$$

and similarly

$$\begin{aligned} (3) \quad \frac{1}{x} + \frac{1}{x + 1} + \dots + \frac{1}{y} \\ &= \log \frac{2y + 1}{2x - 1} - \sum_{k=x}^y \sum_{j=1}^{\infty} \frac{2}{(2j + 1)(2k)^{2j+1}}. \end{aligned}$$

Now (1) and our assumption that  $n < x$  imply that for any  $j \geq 1$ ,

$$\sum_k \frac{2}{(2j + 1)(2k)^{2j+1}}$$

is greater when summed over  $k = m, m + 1, \dots, n$  than over  $k = x, x + 1, \dots, y$  and so, comparing the right sides of (2) and (3), we see that

$$(2n + 1)/(2m - 1) > (2y + 1)/(2x - 1).$$

Thus, ignoring the sum on the right side of (3), we may write

$$(4) \quad \log \frac{(2n + 1)(2x - 1)}{(2m - 1)(2y + 1)} < \sum_{k=m}^n \sum_{j=1}^{\infty} \frac{2}{(2j + 1)(2k)^{2j+1}}.$$

The infinite sum on the right can be replaced by  $4/3$  times the first term, since each term is more than 4 times the next. The numerator of the fraction on the left exceeds the denominator by at least 2, since both are odd, and hence the left side exceeds

$$\log \left( 1 + \frac{2}{(2m - 1)(2y + 1)} \right) > \frac{1}{(2m - 1)(2y + 1)}.$$

Thus we have

$$(5) \quad \frac{1}{(2m-1)(2y+1)} < \sum_{k=m}^n \frac{2 \cdot 4/3}{3(2k)^3} < \frac{1}{9m^2} \sum_{k=m}^n \frac{1}{k} = \frac{1}{9m^2} \sum_{k=x}^y \frac{1}{k}.$$

But the last sum has fewer than  $x^{1/2}$  terms (by the lemma) and each term is not greater than  $1/x$ . And since  $(2m-1)(2y+1) < 4my$ , inequality (5) implies that

$$\frac{1}{4my} < \frac{1}{9m^2} \cdot \frac{x^{1/2}}{x}$$

or

$$(6) \quad 9mx^{1/2} < 4y.$$

But also  $1/m \leq 1/m + \dots + 1/n < x^{1/2} \cdot (1/x) = 1/x^{1/2}$ , so that  $x^{1/2} < m$ , which together with (6) implies that  $9x < 4y$ , which contradicts the lemma. This completes the proof of Theorem 2.

In conclusion, we observe that  $1/2 + 1/3 + 1/4 \equiv 1/12 \pmod{1}$ . Whether the sums in equation (1) are congruent  $\pmod{1}$  for infinitely many values  $m, n, x, y$  is an unsolved problem.

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