conjugates of $\alpha$ is in $\mathfrak{P}_i$, $i=1, \cdots, m$; hence neither is any power of their product. Some such power, however, is in $\mathfrak{R}$, hence in $\mathfrak{P}\cap\mathfrak{R}\subset\mathfrak{P}_1$. This is a contradiction and completes the proof.

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# NOTE ON AN ASYMMETRIC DIOPHANTINE APPROXIMATION

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1. **Introduction.** In a recent paper B. Segre [1][1] introduced a new type of Diophantine approximation which he called *asymmetric*, since the intervals of approximation are divided into two partial intervals which are in an arbitrarily given ratio. His main result is the following theorem [1, p. 357]:

THEOREM 1. *Every irrational $\theta$ has an infinity of rational approximations $x/y$ such that*

$$\frac{-1}{y^2(1+4\tau)^{1/2}} < \frac{x}{y} - \theta < \frac{\tau}{y^2(1+4\tau)^{1/2}} \qquad (y>0), \tag{1}$$

*where $\tau$ is any given non-negative real number.*

This theorem is classic for $\tau=0$, cf. [2, p. 139], and for $\tau=1$ it reduces to the fundamental result due to Hurwitz [2, p. 163]. No other particular cases of the theorem seem to be known.

Segre's proof of (1) is geometrical. The purpose of this note is to show that when $\tau\geqq 1$ it is possible to give a very simple arithmetical proof. The method is a generalization of that used by Khintchine [3] for the special case when $\tau=1$.

2. **Proof of Theorem 1.** We suppose that $\theta$ is irrational and that $0<\theta<1$. For an arbitrary positive integer $n$ form the Farey series[2] of order $n$, that is, the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed $n$. Let $a/b$ and $a'/b'$ be the two successive terms of this series which satisfy the inequalities $a/b<\theta<a'/b'$. We distinguish two cases.

Received by the editors October 4, 1945.

[1] Numbers in brackets refer to the references.

[2] See Hardy and Wright [2, p. 23].

*Case* 1. Suppose that $\tau>0$, and that $b/b'>(\xi+1)/2\tau$, or $b/b'<(\xi-1)/2\tau$, where $\xi=(1+4\tau)^{1/2}$. Then setting $\omega=b/b'$ we see that

$$\frac{1}{\xi}\left(\tau+\frac{1}{\omega^2}\right)-\frac{1}{\omega}=\frac{\tau}{\xi\omega^2}\left(\omega-\frac{\xi+1}{2\tau}\right)\left(\omega-\frac{\xi-1}{2\tau}\right)>0; \tag{2}$$

consequently, since $a'b-ab'=1$,

$$\frac{a'}{b'}-\frac{a}{b}=\frac{1}{b'^2\omega}<\frac{1}{b'^2\xi}\left(\tau+\frac{1}{\omega^2}\right)=\frac{\tau}{b'^2\xi}+\frac{1}{b^2\xi}, \tag{3}$$

which implies that

$$\frac{a}{b}+\frac{1}{b^2\xi}>\frac{a'}{b'}-\frac{\tau}{b'^2\xi}\cdot \tag{4}$$

Hence $\theta$ must be interior to one or the other of the intervals

$$\left(\frac{a}{b},\ \frac{a}{b}+\frac{1}{b^2\xi}\right)\quad\text{or}\quad\left(\frac{a'}{b'}-\frac{\tau}{b'^2\xi},\ \frac{a'}{b'}\right). \tag{5}$$

Then, according as $\theta$ belongs to the first or to the second interval, we have, respectively, the inequalities

$$-1/b^2\xi<a/b-\theta<0,\quad\text{or}\quad 0<a'/b'-\theta<\tau/b'^2\xi.$$

Thus (1) is true, where for $y$ we take either $b$ or $b'$. The infinity of solutions is assured since, for irrational $\theta$, $b$ and $b'$ increase as $n$ increases.

If $\omega=b/b'=(\xi\pm1)/2\tau$, then both $\tau$ and $\xi$ must be rational. Then the inequality sign in (4) is replaced by an equality sign, and the right and left end points of the intervals in (5) coincide. But $\theta$, being irrational, cannot be equal to this common end point. Hence $\theta$ must be interior to one or the other of these intervals, and the proof proceeds as already explained.

*Case* 2. We now suppose that $(\xi-1)/2\tau<b/b'<(\xi+1)/2\tau$. We consider in turn the two sub-intervals

$$\left(\frac{a}{b},\ \frac{a+a'}{b+b'}\right)\quad\text{and}\quad\left(\frac{a+a'}{b+b'},\ \frac{a'}{b'}\right).$$

For the first sub-interval, write $\omega=b/(b+b')$. Then it is easy to see that $\omega<(\xi-1)/2\tau$, $\tau>0$, and that the inequality (2) is again true. Hence

$$\frac{a+a'}{b+b'}-\frac{a}{b}=\frac{1}{(b+b')^2\omega}<\frac{1}{b^2\xi}+\frac{\tau}{(b+b')^2\xi},$$

and, proceeding as in Case 1, we see that (1) is true where for $y$ we take either $b$ or $b+b'$.

The second sub-interval is handled in exactly the same way. We set $\omega=(b+b')/b'$, then $\omega>(\xi+1)/2\tau$, provided $\tau\geqq 1$. This is the first time we need this restriction on $\tau$. The inequality (1) is again true where $y$ is either $b'$ or $b+b'$.

## References

1 B. Segre, *Lattice points in infinite domains, and asymmetric Diophantine approximations*, Duke Math. J. vol. 12 (1945) pp. 337–365.

2. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford, 1938.

3. A. Khintchine, *Neuer Beweis und Verallgemeinerung eines Hurwitzschen Satzes*, Math. Ann. vol. 111 (1935) pp. 631–637.

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