A NOTE ON THE RIEMANN ZETA-FUNCTION

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Let \( \rho = \beta + i\gamma \) be the zeros of the Riemann zeta-function \( \zeta(1/2 + z) \) whose real part \( \beta \geq 0 \). Then we have the following formula which is an improvement on Paley-Wiener's \([1, \text{p. 78}]\)

\[
\int_1^T \frac{\log \left| \zeta(1/2 + it) \right|}{t^2} \, dt = 2\pi \sum_{\nu=1}^{\infty} \frac{\beta_\nu}{|\rho_\nu|^2} + \int_0^{\pi/2} R \{ \exp \{ \log \zeta(1/2 + e^{i\theta}) \} \} \, d\theta + O\left( \frac{\log T}{T} \right).
\]

In order to prove this formula let \( \rho_\nu (\nu = 1, 2, \ldots, n) \) be the \( n \) zeros of \( \zeta(1/2 + z) \) for which \( 0 < \gamma_\nu < T \) and \( 0 \leq \beta_\nu < 1/2 \). We require the following lemma:

**Lemma.** Let \( K \) be the unit semicircle with center \( z = 0 \) lying in the right half-plane \( R(z) > 0 \) and let \( C \) be the broken line consisting of three segments \( L_1 \) (0 \( \leq \) \( x \) \( \leq \) \( T \), \( y = T \)), \( L_2 \) (0 \( \leq \) \( x \) \( \leq \) \( T \), \( y = -T \)) and \( L_3 \) \((x = T, -T \leq y \leq T)\). Then

\[
\frac{1}{\pi} \int_{C} \log \left| \zeta(1/2 + iz) \right| \frac{1}{z^2} \, dz = 2 \sum_{\nu=1}^{n} \frac{\beta_\nu}{|\rho_\nu|^2} + \frac{1}{2\pi i} \int_{K} \frac{1}{z} \frac{\log \zeta(1/2 + z)}{z^2} \, dz - \frac{1}{2\pi i} \int_{C} \frac{\log \zeta(1/2 + z)}{z^2} \, dz.
\]

This is a form of Carleman's theorem which can be proved by a method of proof analogous to that of Littlewood's theorem (Titchmarsh \([3, \text{pp. 130-134}]\)).

Let \( \Gamma \) be a contour describing \( C, K \) and the corresponding part of the imaginary axis, and let \( \rho \) be a point interior to \( \Gamma \), and \( \log(z - \rho) \) be taken as its principal value. We write \( C_1 \) as a contour describing \( \Gamma \) in positive direction to the point \( i\gamma \), then along the segment \( y = \gamma, 0 < x < \beta - r \), and describing a small circle with center \( z = \rho \), radius \( r \), then going back along the negative side of this segment to \( i\gamma \), and then along \( \Gamma \) to the starting point.

By Cauchy's theorem we get

\[
\int_{C_1} \frac{\log(z - \rho)}{z^2} \, dz = 0.
\]

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\(^1\) Numbers in brackets refer to the references cited at the end of the paper.
Hence
\[ \frac{1}{2\pi i} \oint_{\gamma} \frac{\log(z - \rho)}{z^2} \, dz = -\int_{0}^{\beta} \frac{dx}{(x + i\gamma)^2} \]
where the integral round the small circle with center \( z = \rho \), radius \( r \), tends to zero as \( r \to 0 \). This formula is also true for \( \beta = 0 \).

Put \( \xi(1/2 + z) = \phi(z) \prod_{n=1}^{\infty} (z - \rho_n) \prod_{n=1}^{\infty} (z + \overline{\rho_n}) \) where \( \phi(z) \) is regular and has no zero in and on \( \Gamma \). Then we get
\[ \frac{1}{2\pi i} \oint_{\gamma} \frac{\log\xi(1/2 + z)}{z^2} \, dz = \sum_{n=1}^{\infty} \frac{\beta_n}{\rho_n} - \frac{1}{i\gamma} \sum_{n=1}^{\infty} \left( \frac{1}{\rho_n} + \frac{1}{i\gamma} \right) \]
\[ = 2 \sum_{n=1}^{\infty} \frac{\beta_n}{|\rho_n|^2} \cdot \]

From this the lemma follows.

Now we have
\[ \int_{c} \frac{\log\xi(1/2 + z)}{z^2} \, dz = -\int_{L_1} + \int_{L_2} + \int_{L_3} . \]

On account of
\[ \log\xi(1/2 + x + iT) = O(1) \quad \text{for } x \geq 1 \]
we have
\[ \int_{L_1} = \int_{0}^{1} \frac{\log\xi(1/2 + x + iT)}{(x + iT)^2} \, dx + O\left(\frac{1}{T}\right) . \]

Since (Titchmarsh [2, p. 5])
\[ \arg\xi(1/2 + x + iT) = O(\log T) \quad \text{for } 0 \leq x \leq 1 \]
and (Titchmarsh [2, p. 59])
\[ \log |\xi(1/2 + x + iT)| \]
\[ = \frac{1}{2} \sum_{|\gamma - T| < 1} \log \left\{ (x - \beta)^2 + (T - \gamma)^2 \right\} + O(\log T) , \]
then
\[ \int_{0}^{1} \frac{\log\xi(1/2 + x + iT)}{(x + iT)^2} \, dx = O\left(\frac{\log T}{T^2}\right) . \]

From (3) and (4) we get
\[ \int_{L_1} = O\left(\frac{\log T}{T}\right) . \]
Similarly

\[ \int_{L_0} = O\left(\frac{\log T}{T}\right). \]

Since \( \log \xi(1/2 + iT + iy) = O(2^{-T}) \), we get

\[ \int_{L_0} = O(T2^{-T}). \]

By (1), (2), (5), (6) and (7) we have

\[ \int_1^T \frac{\log |\xi(1/2 + it)|}{t^2} \, dt = 2\pi \sum_{\gamma > T} \frac{\beta_\gamma}{|\rho_\gamma|^2} + \frac{1}{2i} \int_K \frac{\log \xi(1/2 + z)}{z^2} \, dz + O\left(\frac{\log T}{T}\right). \]

But (Ingham [4, p. 70])

\[ \sum_{\gamma > T} \frac{\beta_\gamma}{|\rho_\gamma|^2} = O\left(\frac{1}{\gamma^2}\right) = O\left(\frac{\log T}{T}\right). \]

The formula follows from (8) and (9).

Finally, if we make \( T \to \infty \) then

\[ \int_1^\infty \frac{\log |\xi(1/2 + it)|}{t^2} \, dt = \int_0^{\pi/2} R \{ e^{-i\theta} \log \xi(1/2 + e^{i\theta}) \} \, d\theta \]
gives a necessary and sufficient condition for the truth of the Riemann hypothesis.

REFERENCES


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