NOTE ON A THEOREM OF MURRAY

GEORGE W. MACKEY

1. **Introduction.** In a recent paper[1] [1][2] Murray has shown that in any reflexive separable Banach space $\mathfrak{B}$ every closed subspace $\mathfrak{M}$ admits what he calls a quasi-complement, that is, a second closed subspace $\mathfrak{N}$ such that $\mathfrak{M}\cap\mathfrak{N}=0$ and such that $\mathfrak{M}\dotplus\mathfrak{N}$, the smallest subspace containing both $\mathfrak{M}$ and $\mathfrak{N}$, is dense in $\mathfrak{B}$. It is the purpose of this note to give a simpler proof of the following somewhat more general theorem.

THEOREM. *Let $\mathfrak{B}$ be a separable normed linear space (not necessarily reflexive or even complete) and let $\mathfrak{M}$ be a closed subspace of $\mathfrak{B}$. Then there exists a second closed subspace $\mathfrak{N}$ such that $\mathfrak{M}\cap\mathfrak{N}=0$ and $\mathfrak{M}\dotplus\mathfrak{N}$ is dense in $\mathfrak{B}$.*

In proving this theorem it is convenient to make use of the notion of closed subspace of a linear system discussed at length in Chapter III of [2]. We repeat the necessary definitions here. A linear system X_L is an abstract linear space X together with a linear subspace L of the space X^* of all linear[3] functionals defined on X. If $l(x)=0$ for all l in L implies that $x=0$ (that is, if L is total) we say that X_L is a regular linear system. If M is a subspace of X [L] we denote by M' the set of all l in L [x in X] such that $l(x)=0$ for all x in X [l in L]. It is clear that $M\subseteq N$ implies $N'\subseteq M'$ and that $M''\supseteq M$. Since $M'''=(M'')'\subseteq M'$ and since $M'''=(M')''\supseteq M'$ it follows that $M'=M'''$ and hence that $M=M''$ if and only if M is of the form N'. A subspace having either and hence both of these properties is said to be closed. We observe that the operation $'$ sets up a one-to-one inclusion inverting correspondence between the closed subspaces of X and L respectively.

2. **Two lemmas.** The proof of the theorem is based essentially on the following lemma.

LEMMA 1. *Let X_L be a regular linear system such that both X and L are $\aleph_0$ dimensional, that is, have $\aleph_0$ independent generators. Then if M*

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[1] The author is indebted to Professor Murray for letting him read this paper in manuscript form and for several helpful discussions during the preparation of the present note.

[2] Numbers in brackets refer to the bibliography.

[3] By linear we mean additive and homogeneous.

is any closed subspace of X_L *there exists a second closed subspace* N *of* X_L *such that* $M \dotplus N = X$ *and* $M' \dotplus N' = L$.

The proof of Lemma 1 is an easy consequence of a second lemma which is proved in its present form on page 171 of [2] and in other forms elsewhere but which we prove again here for completeness.

LEMMA 2. *Let* X_L *be as in Lemma* 1. *Then there exist sequences of elements* $x_1, x_2, \cdots$ *and* $l_1, l_2, \cdots$ *of* X *and* L *respectively such that* $x_1 \dotplus x_2 \dotplus \cdots = X, l_1 \dotplus l_2 \dotplus \cdots = L$ *and* $l_i(x_j) = \delta_i^j$ *for* $i, j = 1, 2, \cdots$.

PROOF. Let $y_1, y_2, \cdots$ and $m_1, m_2, \cdots$ generate X and L respectively. We define $x_1, x_2, \cdots$ and $l_1, l_2, \cdots$ by induction. Let $l_1 = m_1$ and let $x_1 = y_{n_1}/m_1(y_{n_1})$ where n_1 is the first integer such that $m_1(y_{n_1}) \neq 0$. Suppose that $x_1, x_2, \cdots, x_k$ and $l_1, l_2, \cdots, l_k$ have been defined. If k is odd let n_0 be the first integer such that $y_{n_0} \notin x_1 \dotplus x_2 \dotplus \cdots \dotplus x_k$ and let $x_{k+1} = y_{n_0} - (l_k(y_{n_0})x_k + \cdots + l_1(y_{n_0})x_1)$. Then let $\bar{n}$ be the first integer such that $m_{\bar{n}}(x_{k+1}) \neq 0$ and let $l_{k+1} = (m_{\bar{n}} - (m_{\bar{n}}(x_k)l_k + \cdots + m_{\bar{n}}(x_1)l_1))/m_{\bar{n}}(x_{k+1})$. If k is even let n_0 be the first integer such that $m_{n_0} \notin l_1 \dotplus l_2 \dotplus \cdots \dotplus l_k$ and let $l_{k+1} = m_{n_0} - (m_{n_0}(x_k)l_k + \cdots + m_{n_0}(x_1)l_1)$. Then let $\bar{n}$ be the first integer such that $l_{k+1}(y_{\bar{n}}) \neq 0$ and let $x_{k+1} = (y_{\bar{n}} - (l_k(y_{\bar{n}})x_k + \cdots + l_1(y_{\bar{n}})x_1))/l_{k+1}(y_{\bar{n}})$. It follows at once by induction that $l_i(x_j) = \delta_i^j$ for $i, j = 1, 2, \cdots$ and it is clear that $X = x_1 \dotplus x_2 \dotplus \cdots$ and $L = l_1 \dotplus l_2 \dotplus \cdots$.

PROOF OF LEMMA 1. For definiteness we shall assume that M and M' are infinite-dimensional. The only difference in the contrary case is that certain infinite sequences must be replaced by finite ones. That Lemma 2 is true when X and L are finite-dimensional is obvious. Applying Lemma 2 to M and the linear functionals on M defined by the members of L we may infer the existence of sequences of elements $x_1, x_2, x_3, \cdots$ and $m_1, m_2, m_3, \cdots$ of M and L respectively such that $x_1 \dotplus x_2 \dotplus \cdots = M$, $M' \dotplus m_1 \dotplus m_2 \dotplus \cdots = L$ and $m_i(x_j) = \delta_i^j$ for $i, j = 1, 2, \cdots$. Similarly by applying Lemma 2 to M' and the linear functionals on M' defined by members of X and remembering that $M'' = M$ we may infer the existence of sequences of elements $f_1, f_2, f_3, \cdots$ and $z_1, z_2, z_3, \cdots$ of M' and X respectively such that $f_1 \dotplus f_2 \dotplus \cdots = M'$, $M \dotplus z_1 \dotplus z_2 \dotplus \cdots = X$ and $f_i(z_j) = \delta_i^j$ for $i, j = 1, 2, \cdots$. Now for each i and $j = 1, 2, \cdots$ let

$$y_j = z_j - (m_1(z_j)x_1 + m_2(z_j)x_2 + \cdots + m_j(z_j)x_j),$$
$$l_i = m_i - (m_i(z_1)f_1 + m_i(z_2)f_2 + \cdots + m_i(z_{i-1})f_{i-1}),$$

where f_0 and z_0 are to be taken as zero. Then keeping in mind the fact that $f_i(z_j)=m_i(x_j)=\delta_i^j$ and $f_i(x_j)=0$ for $i, j=1, 2, \cdots$ it is easy to verify that $l_i(y_j)=0$ and $f_i(y_j)=l_i(x_j)=\delta_i^j$. The statement of the lemma now follows at once on setting $N=y_1+y_2+\cdots$. In fact since $x_1+x_2+\cdots+y_1+y_2+\cdots=x_1+x_2+\cdots+z_1+z_2+\cdots=M+z_1+z_2+\cdots=X$ and $l_1+l_2+\cdots+f_1+f_2+\cdots=l_1+l_2+\cdots+m_1+m_2+\cdots=M'+m_1+m_2+\cdots=L$, it is easy to see that $N'=l_1+l_2+\cdots$ and $N''=y_1+y_2+\cdots$. Thus $N''=N$ so that N is closed and $M+N=X$ while $M'+N'=L$.

3. **Proof of the theorem.** Since $\mathfrak{B}$ is separable it is clear that $\mathfrak{B}|\mathfrak{M}$ is also.[4] It follows then from Théorème 4 on page 124 of [3] that there exists a countable total set of members of the conjugate of $\mathfrak{B}|\mathfrak{M}$ and a countable total set of members of the conjugate of $\mathfrak{B}$. Now every member of the conjugate of $\mathfrak{B}|\mathfrak{M}$ has associated with it in an obvious fashion a member of the conjugate of $\mathfrak{B}$ which vanishes throughout $\mathfrak{M}$. Thus the first countable total set defines a countable set of elements of the conjugate of $\mathfrak{B}$ the intersections of the null spaces of which is $\mathfrak{M}$. Denote the linear span of these two countable subsets of the conjugate of $\mathfrak{B}$ by L. Since $\mathfrak{M}$ and $\mathfrak{B}$ are separable there exists a dense countable set in $\mathfrak{B}$ a subset of which is a dense set in $\mathfrak{M}$. Let X be the linear span of this countable set and let $M=\mathfrak{M}\cap X$. It is obvious that the X, L and M so defined satisfy the hypotheses of Lemma 1 and that the closures of M and X are $\mathfrak{M}$ and $\mathfrak{B}$ respectively. That M is closed as a subspace of the linear system X_L follows from the fact that $\mathfrak{M}$ is an intersection of null spaces of members of L. Let N be the closed subspace of X_L whose existence is guaranteed by Lemma 1. We define $\mathfrak{N}$ as the closure (in $\mathfrak{B}$) of N. Since $\mathfrak{M}+\mathfrak{N}\supseteq M+N=X$ it is clear that $\mathfrak{M}+\mathfrak{N}$ is dense in $\mathfrak{B}$. Suppose that $x\in\mathfrak{M}\cap\mathfrak{N}$. Since $\mathfrak{M}$ is the closure of M every element in M' vanishes throughout $\mathfrak{M}$. Similarly every element in N' vanishes throughout $\mathfrak{N}$. Thus every element in $M'+N'=L$ vanishes on x. But L is total. Therefore $x=0$. Thus $\mathfrak{M}\cap\mathfrak{N}=0$ and the theorem is proved.

4. **Remarks.** Murray's paper closes with a proof that in reflexive spaces quasi-complements which are not at the same time complements[5] are very non-unique in the sense that every such both properly contains and is properly contained in other quasi-complements. This theorem and its proof may be extended to the nonreflexive case (but not the incomplete one) by considering the closed subspaces of the linear systems $\mathfrak{B}_A$ and $\mathfrak{A}_B$ where $\mathfrak{A}$ is the conjugate of $\mathfrak{B}$ rather than those of the Banach spaces $\mathfrak{A}$ and $\mathfrak{B}$. The needed fact that the linear

[4] Here $\mathfrak{B}|\mathfrak{M}$ denotes the quotient or difference space of $\mathfrak{B}$ mod $\mathfrak{M}$.

[5] Complements of course are not unique either.

union of a closed subspace with a finite-dimensional one is again closed follows from Theorem III-1 of [2]. The trouble when $\mathfrak{B}$ is not complete is that $\mathfrak{M}'$ and $\mathfrak{N}'$ may be complementary even when $\mathfrak{M}$ and $\mathfrak{N}$ are not. That this can indeed happen is shown by the first example at the bottom of page 173 of [2].

Using this same device one may carry over Murray's theory of the connection between quasi-complements and closed projections almost word for word to the nonreflexive and noncomplete case.

Making further use of the methods and theorems of Chapter III of [2] one can show that if $\mathfrak{B}$ is separable and $\mathfrak{M}$ is a closed subspace of $\mathfrak{B}$ such that neither $\mathfrak{M}$ nor $\mathfrak{B} | \mathfrak{M}$ is finite-dimensional then the quasi-complement of $\mathfrak{M}$ can always be selected so as not to be a complement and that whenever $\mathfrak{B}$ is complete and $\mathfrak{M}$ and $\mathfrak{N}$ are quasi-complementary and not complementary then $\mathfrak{B}$ has infinitely many linearly independent elements mod $\mathfrak{M} \dotplus \mathfrak{N}$.

Bibliography

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Harvard University