ON PROXIMATE ORDERS OF INTEGRAL FUNCTIONS

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Let $f(z)$ be an integral function of finite order ρ and let $M(r) = \max_{|z|=r}|f(z)|$. It is possible to find[1] a positive continuous function $\rho(r)$ having the following properties.

(1) $\rho(r)$ is differentiable for $r > r_0$ except at isolated points at which $\rho'(r-0)$ and $\rho'(r+0)$ exist;

$$\lim_{r\to\infty} \rho(r) = \rho; \tag{2}$$

$$\lim_{r\to\infty} r\rho'(r) \log r = 0; \tag{3}$$

$$\limsup_{r\to\infty} \frac{\log M(r)}{r^{\rho(r)}} = 1. \tag{4}$$

Such a function is called a Lindelöf's proximate order for the integral function $f(z)$. The proof given by Valiron for the existence of proximate orders is based on some rather deep results due to Blumenthal. The object of this note is to give a particularly simple proof of the existence of proximate orders. The proof given here makes no use of the special properties of $M(r)$ and is therefore of wider scope.

Let $\sigma(r) = \log \log M(r)/\log r$. Either (A) $\sigma(r) > \rho$ for a sequence of values of r tending to infinity, or (B) $\sigma(r) \leqq \rho$ for all large r.

In case (A) we define $\phi(r) = \max_{x \geqq r}\{\sigma(x)\}$. Since $\sigma(r)$ is continuous, $\limsup_{r\to\infty} \sigma(r) = \rho$ and $\sigma(r) > \rho$ for a sequence of values of r tending to infinity. Therefore $\phi(r)$ exists. $\phi(r)$ is a nonincreasing function of r.

Let $r_1 > e^{e^e}$ and $\phi(r_1) = \sigma(r_1)$. Such values will exist for a sequence of values of r tending to infinity.

Let $\rho(r_1) = \phi(r_1)$. Let t_1 be the smallest integer not less than $1 + r_1$ such that $\phi(r_1) > \phi(t_1)$, and let $\rho(r) = \rho(r_1) = \phi(r_1)$ for $r_1 < r \leqq t_1$.

Define u_1 as follows:

$$u_1 > t_1,$$

$$\rho(r) = \rho(r_1) - \log\log\log r + \log\log\log t_1 \qquad \text{for } t_1 \leqq r \leqq u_1,$$

$$\rho(r) = \phi(r) \qquad \text{for } r = u_1,$$

but $\rho(r) > \phi(r)$ for $t_1 \leqq r < u_1$. Let r_2 be the smallest value of r for which $r_2 \geqq u_1$ and

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[1] G. Valiron, *Lectures on the general theory of integral functions*, Cambridge, 1923, pp. 64–67.

$$\phi(r_2) = \sigma(r_2).$$

If $r_2 > u_1$, then let $\rho(r) = \phi(r)$ for $u_1 \leqq r \leqq r_2$. Since $\phi(r)$ is constant for $u_1 \leqq r \leqq r_2$, therefore $\rho(r)$ is constant for $u_1 \leqq r \leqq r_2$. We repeat the argument and obtain that $\rho(r)$ is differentiable in adjacent intervals. Further

$$\rho'(r) = 0 \quad \text{or} \quad \frac{-1}{r \log r \log \log r},$$

and $\rho(r) \geqq \phi(r) \geqq \sigma(r)$, for all $r \geqq r_1$. Further $\rho(r) = \sigma(r)$ for an infinity of values of $r = r_1, r_2, \cdots$, $\rho(r)$ is nonincreasing and $\lim_{r\to\infty} \phi(r) = \rho$. Hence

$$\limsup_{r\to\infty} \rho(r) = \lim_{r\to\infty} \rho(r) = \rho,$$

and since $\log M(r) = r^{\sigma(r)} = r^{\rho(r)}$ for an infinity of r, $\log M(r) < r^{\rho(r)}$ for the remaining r, therefore $\limsup_{r\to\infty} \log M(r)/r^{\rho(r)} = 1$.

Let $\sigma(r) \leqq \rho$ for all large r (case (B)). Here there are two possibilities:

$$\text{(B.1)} \qquad \sigma(r) = \rho$$

for at least a sequence of values of r tending to infinity;

$$\text{(B.2)} \qquad \sigma(r) < \rho$$

for all large values of r.

In case (B.1) we take $\rho(r) = \rho$ for all values of r.

In case (B.2) let $\xi(r) = \max_{X \leqq x \leqq r} \{\sigma(x)\}$ where $X > e^{e^e}$ is such that $\sigma(x) < \rho$ whenever $x \geqq X$. $\xi(r)$ is nondecreasing. Take a suitably large value of $r_1 > X$ and let

$$\rho(r_1) = \rho,$$
$$\rho(r) = \rho + \log\log\log r - \log\log\log r_1 \qquad \text{for } s_1 \leqq r \leqq r_1,$$

where $s_1 < r_1$ is such that $\xi(s_1) = \rho(s_1)$. If $\xi(s_1) \neq \sigma(s_1)$, then we take $\rho(r) = \xi(r)$ up to the nearest point $t_1 < s_1$ at which $\xi(t_1) = \sigma(t_1)$. $\rho(r)$ is then constant for $t_1 \leqq r \leqq s_1$.

If $\xi(s_1) = \sigma(s_1)$, then let $t_1 = s_1$.

Choose $r_2 > r_1$ suitably large and let $\rho(r_2) = \rho$,

$$\rho(r) = \rho + \log\log\log r - \log\log\log r_2 \qquad \text{for } s_2 \leqq r \leqq r_2$$

where s_2 ($< r_2$) is such that[2] $\xi(s_2) = \rho(s_2)$. If $\xi(s_2) \neq \sigma(s_2)$, then let

[2] s_2 is given by the largest positive root of $\xi(s_2) = \rho - \log\log\log r_2 + \log\log\log s_2$.

$\rho(r)=\xi(r)$ for $t_2 \leqq r \leqq s_2$ where t_2 $(<s_2)$ is the point nearest to s_2 at which $\xi(t_2)=\sigma(t_2)$. If $\xi(s_2)=\sigma(s_2)$, then let $t_2=s_2$. For $r<t_2$ let $\rho(r)=\rho(t_2)+\log\log\log t_2-\log\log\log r$ for $u_1 \leqq r \leqq t_2$ where u_1 $(<t_2)$ is the point of intersection of $y=\rho$ with $y=\rho(t_2)+\log\log\log t_2-\log\log\log r$.

Let $\rho(r)=\rho$ for $r_1 \leqq r \leqq u_1$. It is always possible to choose r_2 so large that $r_1<u_1$. We repeat the procedure and note that

$$\rho(r) \geqq \xi(r) \geqq \sigma(r)$$

and $\rho(r)=\sigma(r)$ for $r=t_1, t_2, t_3, \cdots$. Hence $\lim_{r\to\infty}\rho(r)=\rho$, and

$$\limsup_{r\to\infty}\frac{\log M(r)}{r^{\rho(r)}}=1.$$

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A NOTE ON THE SPECTRAL THEOREM

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1. Introduction. Although the connections between the spectral resolution of a self-adjoint transformation in Hilbert space, the moment problem, and Riesz' integral representation [1][1] for linear functionals on the space C are known (cf. Stone [2], Murray [3], Widder [4], Lengyel [5]), the following elementary derivation of the spectral theorem from the Riesz theorem exhibits the connections in, perhaps, the simplest light. We consider only *bounded* self-adjoint transformations H; one can treat an unbounded H by considering $(I+H^2)^{-1}$, which is bounded and self-adjoint [3, p. 95]. Note that the derivation does not involve the separability of the Hilbert space $\mathfrak{H}$.

2. Six lemmas. Let H be a self-adjoint transformation with the bounds a, b—that is, $a\|f\|^2 \leqq (Hf, f) \leqq b\|f\|^2$ for all $f \in H$, and $\|H\| = \max(|a|, |b|)$. Denote by C the space of continuous real-valued functions defined on the closed interval (a, b), with $\|f(x)\| = \max|f(x)|$ $(a \leqq x \leqq b)$. Let $p(x)=\sum_0^n c_i x^i$ be any polynomial with real coefficients, and let $p(H)$ be the corresponding transformation $p(H)=\sum_0^n c_i H^i$, where $H^0=I$.

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[1] Numbers in brackets refer to the references cited at the end of the paper.