# APPROXIMATION IN THE SENSE OF LEAST $p$TH POWERS WITH A SINGLE AUXILIARY CONDITION OF INTERPOLATION

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**Introduction.** Let $w=g(z)$ map the interior of the analytic Jordan curve $C$ conformally into the interior of the circle $|w|=1$. We shall say that the function $f(z)$, analytic interior to $C$, is of class $E_p$ there if $\int_{C_r}|f(z)|^p|dz|$ is bounded for $r<1$, where $C_r$ is the curve $|g(z)|=r$. A function is of class $H_p$ if it is analytic for $|z|<1$, and of class $E_p$ there. We are taking $p>0$.

Of the functions $f(z)$, analytic interior to $C$, of class $E_p$ there, with $f(\alpha)=A$ ($z=\alpha$ a point interior to $C$, $A$ an arbitrary constant), let $F_0(z)$ be the one[1] which minimizes the integral $\int_C|f(z)|^p|dz|$. Let $P_n(z)$ be the minimizing polynomial of degree $n$ with $P_n(\alpha)=A$, for this integral. We shall prove that the sequence $P_n(z)$, $n=0, 1, 2, \cdots$, *converges maximally* to $F_0(z)$ on the closed set $\Gamma$, consisting of $C$ and its interior, and then derive some extensions. We use the term *maximal convergence* in the sense of J. L. Walsh [**3**, p. 80].[2]

We denote by $\rho$ the maximum value of $R$ such that the minimizing function can be extended so as to be analytic and single-valued interior to $\Gamma_R$, as used by Walsh [**3**, p. 80].

**1. Inequalities: unit circle.** We shall start with the results for the unit circle.

THEOREM 1.1. *Of the functions $f(z)$ of class $H_p$ ($p>0$) interior to $C$: $|z|=1$, with $f(\alpha)=A$, $|\alpha|<1$, the one which minimizes the integral $\int_C|f(z)|^p|dz|$ is given by $F_0(z)=A[(|\alpha|^2-1)/(\bar{\alpha}z-1)]^{2/p}$, with the branch for which $F_0(\alpha)=A$.*

For, it is true that $\int_C|f(z)|^p|dz|\geqq 2\pi|f(0)|^p$, so that for the case $\alpha=0$, the minimizing function is $F_0(z)=A$. If, in the general case, we map the interior of the unit circle conformally into itself, with $z=\alpha$ corresponding to the origin, the desired result is obtained.

The main new tool exhibited by the paper is the following theorem.

THEOREM 1.2. *Let $f_n(z)$ be a sequence of functions of class $H_p$ with $f_n(\alpha)=A$ and $\int_C|f_n(z)|^p|dz|\leqq\int_C|F_0(z)|^p|dz|+\epsilon_n$, where $\epsilon_n\to 0$ as*

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[1] The minimizing function $F_0(z)$ is unique.

[2] Numbers in brackets refer to the references cited at the end of the paper.

$n \to \infty$. *If $D$ is any closed region lying entirely interior to $C$, there is a constant $M$, depending only on $D$, such that*

$$|f_n(z) - F_0(z)| \leqq M\epsilon_n^{1/2}, \qquad \textit{for } z \textit{ in } D.$$

This implies continuous convergence of $f_n(z)$ to $F_0(z)$ interior to $C$, and gives a measure of the degree of convergence.

We start the proof with the case $\alpha = 0$, $A = 1$. Then

$$\int_C |f_n(z)|^p |dz| \leqq 2\pi + \epsilon_n, \qquad f_n(0) = 1.$$

With the further restriction that $f_n(z) \neq 0$ for $|z| < 1$, the functions $f_n(z)^{p/2}$ are of class $H_2$ and have the expansions $1 + \sum_{i=1}^{i=\infty} a_{i,n} z^i$, with $\sum_{i=1}^{i=\infty} |a_{i,n}|^2$ convergent, from which, by Parseval's Theorem

$$\int_C |f_n(z)^{p/2}|^2 |dz| = 2\pi \left[ 1 + \sum_{i=1}^{\infty} |a_{i,n}|^2 \right] \leqq 2\pi + \epsilon_n,$$

so that we get the inequality in

$$\int_C |f_n(z)^{p/2} - 1|^2 |dz| = 2\pi \sum_{i=1}^{\infty} |a_{i,n}|^2 \leqq \epsilon_n.$$

By a lemma of J. L. Walsh [**3**, p. 101] this implies, for some constant $M'$,

$$|f_n(z)^{p/2} - 1| \leqq M' \epsilon_n^{1/2}, \qquad z \text{ in } D,$$

where $M'$ depends only on $D$, and we can write

$$f_n(z) - 1 = [1 + \eta_n(z)]^{2/p} - 1, \qquad |\eta_n(z)| \leqq M' \epsilon_n^{1/2}, \qquad z \text{ in } D.$$

For $n$ sufficiently large, the binomial expansion may be used, and there is a constant $M''$ such that, for some integer $N$,

$$|f_n(z) - 1| \leqq M'' \cdot \max_{z \text{ in } D} |\eta_n(z)|, \qquad n > N.$$

This gives the desired inequality for $n$ sufficiently large, and a suitable choice of the constant makes it valid for all $n$.

If now the functions $f_n(z)$ do vanish interior to $C$, let $B_n(z)$ be the Blaschke product for the zeros of $f_n(z)$ interior to $C$, normalized so that $B_n(0) > 0$. We write

$$f_n(z) = B_n(z) \cdot \psi_n(z) \tag{1}$$

where $\psi_n(z) \neq 0$ interior to $C$, and $\psi_n(z)$ is of class $H_p$ for each $n$. Since

$|B_n(z)|$ has boundary values equal to unity a.e. on $C$,

$$\int_C |f_n(z)|^p |dz| = \int_C |\psi_n(z)|^p |dz| \leqq 2\pi + \epsilon_n,$$

from which, setting $\Psi_n(z) = \psi_n(z)/\psi_n(0)$,

$$\int_C |\Psi_n(z)|^p |dz| \leqq (2\pi + \epsilon_n)/[\psi_n(0)]^p. \tag{2}$$

From (1) we have $\psi_n(0) \geqq 1$ and hence from (2), $\int_C |\Psi_n(z)|^p |dz| \leqq 2\pi + \epsilon_n$. The first part of the proof now applies to $\Psi_n(z)$ and gives, for some constant $M'$,

$$|\Psi_n(z)^{p/2} - 1| \leqq M' \epsilon_n^{1/2}, \qquad z \text{ in } D. \tag{3}$$

Since the left member of (2) is not less than $2\pi$, we have

$$[\psi_n(0)]^{p/2} \leqq (1 + \epsilon_n/2\pi)^{1/2} \tag{4}$$

which, with (3), means that for some $M''$

$$|\psi_n(z)^{p/2} - 1| \leqq M'' \cdot \epsilon_n^{1/2}, \qquad z \text{ in } D,$$

and, as in the first part of the proof,

$$|\psi_n(z) - 1| \leqq M \epsilon_n^{1/2}, \qquad z \text{ in } D. \tag{5}$$

We now desire a measure of the degree of convergence of $B_n(z)$. We have a.e. on $C$,

$$|B_n(z) - 1|^2 = |B_n(z)|^2 + 1 - 2\Re[B_n(z)],$$

so that, since $B_n(0)$ is real, and the Cauchy integral formula applies to $B_n(z)$ on $C$,

$$\int_C |B_n(z) - 1|^2 |dz| = 4\pi[1 - B_n(0)].$$

By (1) and (4), $B_n(0) \geqq (1 + \epsilon_n/2\pi)^{-1/p}$, so that for some constant $M'$,

$$\int_C |B_n(z) - 1|^2 |dz| \leqq M' \epsilon_n, \quad n = 0, 1, 2, \cdots,$$

and by the lemma used previously, we have for some $M$,

$$|B_n(z) - 1| \leqq M \epsilon_n^{1/2}, \qquad z \text{ in } D. \tag{6}$$

In (5) and (6), $M$ is not necessarily the same, but the two inequalities imply the existence of a constant $M$, depending only on the region $D$, such that $|f_n(z)-1| \leqq M\epsilon_n^{1/2}$, $z$ in $D$.

It is clear that this inequality holds in the case $f_n(0)=A$, where the convergence is to the function identically equal to $A$.

Let us now return to the original problem, where

$$f_n(\alpha) = A, \qquad F_0(z) = A[(|\alpha|^2 - 1)/(\bar{\alpha}z - 1)]^{2/p}.$$

Here we have

$$\int_C |f_n(z)|^p |dz| \leqq \int_C |F_0(z)|^p |dz| + \epsilon_n. \tag{7}$$

If we make the transformation $w=(z-\alpha)/(1-\bar{\alpha}z)$, (7) becomes

$$\int_{C'} |F_n(w)|^p |dw| \leqq 2\pi |A|^p + \epsilon_n/(1 - |\alpha|^2), \qquad C': |w| = 1,$$

where

$$F_n(w) = (1 + \bar{\alpha}w)^{-2/p}\cdot f_n[(w + \alpha)/(1 + \alpha w)]$$

and since $F_n(0)=A$, this is precisely the situation already treated. The closed region $D$ in the $z$-plane corresponds in the $w$-plane to a closed region $D'$ which lies completely interior to $C'$. Hence, for some constant $M'$,

$$|F_n(w) - A| \leqq M'\epsilon_n^{1/2}, \qquad w \text{ in } D',$$

and we have finally for some constant $M$,

$$|f_n(z) - F_0(z)| \leqq M\epsilon_n^{1/2}, \qquad z \text{ in } D.$$

Continuous convergence interior to $C$ for the case $f_n(0)=1$ was proved by Keldysch and Lavrentieff [1, p. 35], but no attempt was made to determine the degree of convergence.

**2. Maximal convergence: unit circle.** We now estimate the $\epsilon_n$ for minimizing polynomials.

THEOREM 2.1. *Let $P_n(z)$ be the polynomial of degree $n$ with $P_n(\alpha)=A$ for which $\int_C|P_n(z)|^p|dz|$ is a minimum. Then, given any number $R$ such that $1<R<\rho$, there exists a constant $M$, not depending on $n$, such that in the inequality*

$$\int_C |P_n(z)|^p |dz| \leqq \int_C |F_0(z)|^p |dz| + \epsilon_n, \qquad C: |z| = 1,$$

*we may take $\epsilon_n \leqq M/R^{2n}$.*

Let $\pi_n(z)$ be the polynomial of degree $n$, of best approximation to $F_0(z)$ on $\Gamma$ in the sense of least maximum modulus, with the condition $\pi_n(\alpha)=A$. The $\pi_n(z)$ converge maximally to $F_0(z)$ on $\Gamma$ [3, §11.2], so that for any $R$ as above there is a constant $M$ such that

$$|\pi_n(z)-F_0(z)| \leqq M/R^n, \qquad z \text{ on } \Gamma.$$

This implies first of all that for $n$ sufficiently large, the $\pi_n(z)$ do not vanish interior to $C$. Let us now write

$$\pi_n(z) = F_0(z) + R_n(z)$$

from which

$$\pi_n(z)^{p/2} = F_0(z)^{p/2}[1+R_n(z)/F_0(z)]^{p/2}, \quad |R_n(z)| \leqq M/R^n, \qquad z \text{ in } \Gamma.$$

Since $F_0(z)$ is bounded from zero on $\Gamma$, the expansion of the binomial is valid for $n$ sufficiently large, and there is a constant $M'$ and an integer $N$ such that

$$|\pi_n(z)^{p/2}-F_0(z)^{p/2}| \leqq M' \cdot \max_{z \text{ in } \Gamma} |R_n(z)|, \qquad z \text{ in } \Gamma, \quad n > N,$$

or

$$|\pi_n(z)^{p/2}-F_0(z)^{p/2}| \leqq M''/R^n, \qquad z \text{ in } \Gamma, \quad n > N. \tag{8}$$

The function $F_0(z)^{p/2}-\pi_n(z)^{p/2}$ is analytic on $\Gamma$ for each $n$ which is large enough, and vanishes for $z=\alpha$. Hence it is orthogonal to $(1-\bar{\alpha}z)^{-1}$ on $C$, by which we get

$$\int_C [F_0(z)^{p/2}-\pi_n(z)^{p/2}]\cdot\overline{F}_0(z)^{p/2}|dz| = 0. \tag{9}$$

Thus,

$$\begin{aligned}\int_C |F_0(z)^{p/2}-\pi_n(z)^{p/2}|^2|dz| &= \int_C [F_0(z)^{p/2}-\pi_n(z)^{p/2}][\overline{F}_0(z)^{p/2}-\bar{\pi}_n(z)^{p/2}]|dz| \\ &= -\int_C F_0(z)^{p/2}\bar{\pi}_n(z)^{p/2}|dz| + \int_C |\pi_n(z)|^p|dz|.\end{aligned} \tag{10}$$

But by (9), $\int_C F_0(z)^{p/2}\bar{\pi}_n(z)^{p/2}|dz| = \int_C |F_0(z)|^p|dz|$, so that (10) becomes

$$\int_C |\pi_n(z)|^p|dz| = \int_C |F_0(z)|^p|dz| + \int_C |F_0(z)^{p/2}-\pi_n(z)^{p/2}|^2|dz|.$$

Applying (8) we have for some $M'''$ and $n$ large enough

$$\int_C |\pi_n(z)|^p |dz| \leqq \int_C |F_0(z)|^p |dz| + M'''/R^{2n}.$$

But since $P_n(z)$ is the minimizing polynomial of degree $n$ for $P_n(\alpha) = A$, this inequality is also true for $P_n(z)$, and all $n$, and the theorem is proved.

THEOREM 2.2. *The sequence of minimizing polynomials $P_n(z)$ converges maximally to $F_0(z)$ on $\Gamma$.*

It must be shown that given $R$, with $1 < R < \rho$, there exists $M$ such that $|P_n(z) - F_0(z)| \leqq M/R^n$, $z$ in $\Gamma$. Choose an $R_1$ such that $R < R_1 < \rho$. We can then find a closed region $D$, lying completely interior to $C$, such that $D_{R_1/R}$ contains $\Gamma$ in its interior [**3**, §2.2, Theorem 2]. By Theorem 2.1 we have

$$\int_C |P_n(z)|^p |dz| \leqq \int_C |F_0(z)|^p |dz| + M'/R_1^{2n}.$$

But then by Theorem 1.2,

$$|P_n(z) - F_0(z)| \leqq M''/R_1^n, \qquad z \text{ in } D,$$

which gives [**3**, §4.7, Theorem 8, corollary]

$$|P_n(z) - F_0(z)| \leqq M'''/R^n, \qquad z \text{ in } D_{R_1/R}.$$

This inequality, being valid on $D_{R_1/R}$, is valid on its subset $\Gamma$, and the theorem is proved.

3. **Maximal convergence: analytic Jordan curve.** Let $C$ be an arbitrary analytic Jordan curve of the $z$-plane, and $z = \alpha$ a point interior to $C$. Let $w = g(z)$, $z = h(w)$ map the interior of $C$ conformally into the interior of $C'$: $|w| = 1$ so that $z = \alpha$ corresponds to $w = 0$. Let $\Gamma$ consist of $C$ and its interior.

THEOREM 3.1. *Among the functions $f(z)$ of class $E_p(p > 0)$ interior to $C$, with $f(\alpha) = A$, the minimum of $\int_C |f(z)|^p |dz|$ occurs for the function $F_0(z) = A[g'(z)/g'(\alpha)]^{1/p}$.*

For, with the transformation $w = g(z)$, we have

$$\int_C |f(z)|^p |dz| = \int_{C'} |f[h(w)] \cdot h'(w)^{1/p}|^p |dw|.$$

The function $f[h(w)] \cdot h'(w)^{1/p}$ is of class $H_p$ interior to $C'$ and takes on the value $A\cdot h'(0)^{1/p}$ at $w=0$. Hence the minimum of the integral on the right occurs when $F_0[h(w)]\cdot h'(w)^{1/p}=A\cdot h'(0)^{1/p}$, which in the $z$-plane gives $F_0(z)$ as above.

This result in a restricted form is due to Julia,[3] and in more general form to Keldysch and Lavrentieff.

The previous methods carry over to the present situation. Thus, suppose we have the inequality of Theorem 1.2, where the $f_n(z)$ are of class $E_p$ interior to $C$, $f_n(\alpha)=A$, and $\epsilon_n\to 0$ as $n\to\infty$. In the $w$-plane this becomes

$$\int_{C'} \left| f_n[h(w)]\cdot h'(w)^{1/p}\right|^p |dw| \leqq \int_{C'} \left| F_0[h(w)]\cdot h'(w)^{1/p}\right|^p |dw| + \epsilon_n.$$

The hypotheses of Theorem 1.2 are satisfied, and a closed region $D$ entirely interior to $C$ corresponds to a closed region $D'$ entirely interior to $C'$; then there is a constant $M$ such that

$$\left| f_n[h(w)]\cdot h'(w)^{1/p} - F_0[h(w)]\cdot h'(w)^{1/p}\right| \leqq M\epsilon_n^{1/2}, \qquad w \text{ in } D'.$$

But $h'(w)^{1/p}$ is bounded from zero on $\Gamma$, so that in the $z$-plane this becomes

$$\left| f_n(z) - F_0(z)\right| \leqq M'\epsilon_n^{1/2}, \qquad z \text{ in } D.$$

THEOREM 3.2. *Theorem* 1.2 *is valid if all statements in it now refer to an analytic Jordan curve.*

In estimating the $\epsilon_n$ for the minimizing polynomials $P_n(z)$, we use polynomials $\pi_n(z)$ of best approximation to $F_0(z)$ in the sense of least maximum modulus on $\Gamma$, subject to $\pi_n(\alpha)=A$. The proofs carry over, and we have the general result as follows.

THEOREM 3.3. *Let $\Gamma$ be a closed region bounded by the analytic Jordan curve $C$, and let $z=\alpha$ be a point interior to $C$. If $P_n(z)$ is the minimizing polynomial of degree $n$, in the sense of least pth powers* ($p>0$), *to the function $f(z)=0$, on $C$, with $P_n(\alpha)=A$, then the sequence $P_n(z)$ converges maximally on $\Gamma$ to the minimizing function $F_0(z)=A[g'(z)/g'(\alpha)]^{1/p}$.*

4. **Approximation to $(z-\alpha)^{-1}$: analytic Jordan curve.** The results already obtained extend to polynomials of best approximation to certain rational functions.

THEOREM 4.1. *Let $\Gamma$ be a closed region bounded by the analytic Jordan curve $C$, and $z=\alpha$ a point interior to $C$. The function of class $E_p$ interior*

[3] Julia, G. *Leçons sur la représentation conforme des aires simplement connexes*, Paris, 1931.

*to C which minimizes the integral*

$$\int_C |(z-\alpha)^{-1} - f(z)|^p |dz|$$

*is given by*

$$F_0(z) = \frac{1}{z-\alpha} - \frac{g'(z)^{1/p}}{g'(\alpha)^{1/p-1}\cdot g(z)}$$

*where $w=g(z)$ is the mapping function of* §3.

It is seen that $F_0(z)$ is analytic at $z=\alpha$ if properly defined there.

THEOREM 4.2. *Let $f_n(z)$ be a sequence of functions of class $E_p$ interior to $C$, for which*

$$\int_C |(z-\alpha)^{-1} - f_n(z)|^p |dz| \leqq \int_C |(z-\alpha)^{-1} - F_0(z)|^p |dz| + \epsilon_n,$$

*where $\epsilon_n \to 0$ as $n\to\infty$. If $D$ is any closed region lying entirely interior to $C$, there is a constant $M$, depending only on $D$, such that*

$$|f_n(z) - F_0(z)| \leqq M\epsilon_n^{1/2}, \qquad z \text{ in } D.$$

As in §3, we may take $\epsilon_n \leqq M/R^{2n}$ for the minimizing polynomials, which enables us to prove the following theorem.

THEOREM 4.3. *The sequence of polynomials $P_n(z)$ of best approximation in the sense of least pth powers to the function $(z-\alpha)^{-1}$ on an analytic Jordan curve $C$ converges maximally on $\Gamma$ to the minimizing function.*

The method of proof for these results is indicated by the following. We can write

$$\int_C |(z-\alpha)^{-1} - f(z)|^p |dz| = \int_C |g(z)\cdot(z-\alpha)^{-1} - g(z)f(z)|^p |dz|.$$

The function within the absolute value signs on the right is now of class $E_p$ interior to $C$, and at the point $z=\alpha$ takes on the value $g'(\alpha)$. If we apply Theorem 3.1, we get $F_0(z)$ as above.

5. **Introduction of a weight function.** We shall merely state a result, derived by methods as above.

THEOREM. *The minimizing function for*

$$\int_C n(z)|f(z)|^p |dz|, \qquad f(\alpha) = A,$$

*where $C$ is an arbitrary analytic Jordan curve, $z=\alpha$ is a point interior to $C$, $f(z)$ is of class $E_p$ interior to $C$, and $n(z)$ is the modulus on $C$ of a function $N(z)$ analytic and nonvanishing in the closed region $\Gamma$, is*

$$F_0(z) = A\left[\frac{N(\alpha)}{N(z)}\cdot\frac{g'(z)}{g'(\alpha)}\right]^{1/p}.$$

*Let $P_n(z)$ be the corresponding minimizing polynomial of degree $n$. Then the sequence $P_n(z)$, $n=0, 1, 2, \cdots$, converges maximally to $F_0(z)$ on $\Gamma$.*

## References

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# NOTE ON THE LOCATION OF THE CRITICAL POINTS OF HARMONIC FUNCTIONS

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The object of this note is to publish the statement of the following theorem.

Theorem I. *In the extended $(x, y)$-plane let $R_0$ be a simply-connected region bounded by a continuum $C_0$ not a single point, and let the disjoint continua $C_1, C_2, \cdots, C_n$ lie interior to $R_0$ and together with $C_0$ bound a subregion $R$ of $R_0$. By means of a conformal map of $R_0$ onto the unit circle we define in $R_0$ non-euclidean lines, the images of arbitrary circles orthogonal to the unit circle. Denote by $\Pi$ the smallest closed non-euclidean convex region in $R_0$ which contains $C_1, C_2, \cdots, C_n$.*

*Let the function $u(x, y)$ be harmonic interior to $R$, continuous in the closure of $R$, with the values zero on $C_0$ and unity on $C_1, C_2, \cdots, C_n$. Then the critical points of $u(x, y)$ in $R$ are $n-1$ in number and lie in $\Pi$.*

Critical points are of course to be counted according to their multiplicities.

A limiting case of Theorem I has already been established:[1] if $f(z)$

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[1] J. L. Walsh, Bull. Amer. Math. Soc. vol. 45 (1939) pp. 462–470; see p. 465. The result was proved later by W. Gontcharoff, C. R. (Doklady) Acad. Sci. URSS. vol. 36 (1942) pp. 39–41.