where C is an arbitrary analytic Jordan curve, $z=\alpha$ is a point interior to C, $f(z)$ is of class E_p interior to C, and $n(z)$ is the modulus on C of a function $N(z)$ analytic and nonvanishing in the closed region Γ, is

$$F_0(z) = A\left[\frac{N(\alpha)}{N(z)}\cdot\frac{g'(z)}{g'(\alpha)}\right]^{1/p}.$$

Let $P_n(z)$ be the corresponding minimizing polynomial of degree n. Then the sequence $P_n(z)$, $n=0, 1, 2, \cdots$, converges maximally to $F_0(z)$ on Γ.

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NOTE ON THE LOCATION OF THE CRITICAL POINTS OF HARMONIC FUNCTIONS

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The object of this note is to publish the statement of the following theorem.

Theorem I. *In the extended (x, y)-plane let R_0 be a simply-connected region bounded by a continuum C_0 not a single point, and let the disjoint continua $C_1, C_2, \cdots, C_n$ lie interior to R_0 and together with C_0 bound a subregion R of R_0. By means of a conformal map of R_0 onto the unit circle we define in R_0 non-euclidean lines, the images of arbitrary circles orthogonal to the unit circle. Denote by Π the smallest closed non-euclidean convex region in R_0 which contains $C_1, C_2, \cdots, C_n$.*

Let the function $u(x, y)$ be harmonic interior to R, continuous in the closure of R, with the values zero on C_0 and unity on $C_1, C_2, \cdots, C_n$. Then the critical points of $u(x, y)$ in R are $n-1$ in number and lie in Π.

Critical points are of course to be counted according to their multiplicities.

A limiting case of Theorem I has already been established:[1] if $f(z)$

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[1] J. L. Walsh, Bull. Amer. Math. Soc. vol. 45 (1939) pp. 462–470; see p. 465. The result was proved later by W. Gontcharoff, C. R. (Doklady) Acad. Sci. URSS. vol. 36 (1942) pp. 39–41.

is an analytic function whose modulus is constant on the boundary of a simply-connected region R, where $f(z)$ is analytic interior to R and continuous in the closure of R, then the zeros of $f'(z)$ in R lie in the smallest non-euclidean convex polygon in R containing the zeros of $f(z)$ in R. Theorem I is readily established by the use of this limiting case, and of methods developed elsewhere by the present writer;[2] details are left to the reader.

Theorem I admits an extension to the case where R_0 is bounded by C_0, and the subregion R of R_0 is bounded by C_0 and by further disjoint continua $C_1, C_2, \cdots, C_m, C_{m+1}, \cdots, C_n$ in R_0; the function $u(x, y)$ is supposed harmonic interior to R, continuous in the closure of R, with the values zero on C_0, unity on $C_1, C_2, \cdots, C_m$, and minus unity on $C_{m+1}, C_{m+2}, \cdots, C_n$; a non-euclidean line Λ in R_0 (if existent) which separates $C_1, C_2, \cdots, C_m$ from $C_{m+1}, C_{m+2}, \cdots, C_n$ cannot pass through a critical point of $u(x, y)$. If a Λ exists, the points of R_0 which do not lie on any such Λ form two disjoint non-euclidean convex point sets in R_0 which are closed with respect to R_0, which contain respectively $C_1, C_2, \cdots, C_m$ and $C_{m+1}, C_{m+2}, \cdots, C_n$, and which together contain all critical points of $u(x, y)$ in R. This extension of Theorem I may likewise be proved from a limiting case already formulated (loc. cit.) for a region R_0 bounded by a circle.

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[2] Proc. Nat. Acad. Sci. U.S.A. vol. 20 (1934) pp. 551–554.