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ON THE SUMMATION OF MULTIPLE FOURIER SERIES. III

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Let \( f(x) = f(x_1, \ldots, x_k) \) be a function of the Lebesgue class \( L \), which is periodic in each of the \( k \)-variables, having the period \( 2\pi \). Let

\[
a_{x_1 \cdots x_k} = \frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(x) \exp \{ -i(\nu_1 x_1 + \cdots + \nu_k x_k) \} \, dx_1 \cdots dx_k,
\]

where \( \{\nu_k\} \) are all integers. Then the series \( \sum a_{x_1 \cdots x_k} \exp i(\nu_1 x_1 + \cdots + \nu_k x_k) \) is called the multiple Fourier series of the function \( f(x) \), and we write

\[
f(x) \sim \sum a_{x_1 \cdots x_k} \exp i(\nu_1 x_1 + \cdots + \nu_k x_k).
\]

Let the numbers \( (\nu_1^2 + \cdots + \nu_k^2) \), when arranged in increasing order of magnitude, be denoted by \( \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \), and let

\[
C_n(x) = \sum a_{x_1 \cdots x_k} \exp i(\nu_1 x_1 + \cdots + \nu_k x_k),
\]

where the sum is taken over all \( \nu_1^2 + \cdots + \nu_k^2 = \lambda_n \).

\[
\phi(x, t) = \sum C_n(x) \exp (-\lambda_n t),
\]

\[
S_R(x) = \sum_{\lambda_n \leq R^2} C_n(x), \quad \lambda_n \leq R^2 < \lambda_{n+1}.
\]

Received by the editors December 7, 1945.

\(^1\) Papers I and II with the same title are to appear in Proc. London Math. Soc.

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Also, let \( R_k(\lambda) \) and \( r_k(\lambda) \) represent respectively the number of solutions of \( \nu_1^2 + \cdots + \nu_k^2 \leq \lambda \) and of \( \nu_1^2 + \cdots + \nu_k^2 = \lambda \).

The object of this note is to study the convergence of multiple Fourier series, when summed up spherically by Bochner's method, that is, of the series \( \sum C_n(x) \). We prove the following results.

**Theorem I.** If
\[
\sum (\nu_1^2 + \cdots + \nu_k^2)^{k/2} |a_{\nu_1} \cdots a_{\nu_k}|^2 < \infty,
\]
then the series \( \sum_{n=0}^\infty C_n \) converges at every point of continuity of \( f(x) \).

**Theorem II.** If
\[
\sum (\nu_1^2 + \cdots + \nu_k^2)^{k/2+\varepsilon} |a_{\nu_1} \cdots a_{\nu_k}|^2 < \infty, \quad \varepsilon > 0,
\]
then the series \( \sum_{n=0}^\infty C_n \) converges absolutely.

The following result of Bochner\(^2\) is used in the proof of the above theorems.

**Lemma.** At a point of continuity of \( f(x) \), \( \phi(x, t) \) tends to a limit as \( t \) tends to zero.

**Proof of Theorem I.** We shall first prove that
\[
(1) \quad \lim_{R \to \infty} S_R(x) = \lim_{t \to 0^+} \phi(x, t),
\]
whenever the limit on the right exists. Next, by the application of the above lemma, we deduce that at a point where \( f(x) \) is continuous, \( \sum C_n(x) \) is convergent.

Now
\[
S_R(x) - \phi(x, t) = \sum_{s=0}^n C_s [1 - \exp (-\lambda_s t)] - \sum_{s=n+1}^\infty C_s \exp (-\lambda_s t)
\]
\[
= \sum J_1 - J_2,
\]
say. We have,
\[
J_1 = \sum_{s=0}^n C_s [1 - \exp (-\lambda_s t)]
\]
\[
= \sum_{s=0}^n [1 - \exp (-\lambda_s t)] \sum a_{\nu_1} \cdots a_{\nu_k} \exp [i(\nu_1 x_1 + \cdots + \nu_k x_k)]
\]
\[
= \sum a_{\nu_1} \cdots a_{\nu_k} \exp i(\nu_1 x_1 + \cdots + \nu_k x_k) [1 - \exp (-\nu_1^2 - \cdots - \nu_k^2) t],
\]
where the third sum runs over \( \lambda_s = \nu_1^2 + \cdots + \nu_k^2 \) and the last sum runs over \( \nu_1^2 + \cdots + \nu_k^2 \leq \lambda_n \), so that

\[
| J_1 | \leq \sum | a_{\nu_1 \cdots \nu_k} \{ 1 - \exp (- \nu_1^2 - \cdots - \nu_k^2 t) \} | \\
\leq \sum (\nu_1^2 + \cdots + \nu_k^2)^{k/2} | a_{\nu_1 \cdots \nu_k} |^2 \\
\times \sum \{ 1 - \exp (- \nu_1^2 - \cdots - \nu_k^2 t) \} (\nu_1^2 + \cdots + \nu_k^2)^{k/2} \\
\leq O(1) \cdot \left[ \sum r_k(\lambda_n) \lambda_s^{2-k/2} \right]^{1/2}
\]

(3)

where the first sum runs over \( \nu_1^2 + \cdots + \nu_k^2 \leq \lambda_n \).

Now,

\[
\sum_{s=0}^{n} r_k(\lambda_n) \lambda_s^{2-k/2} = \sum_{s=0}^{n-1} R_k(\lambda_n) \{ \lambda_s^{2-k/2} - \lambda_{s+1}^{2-k/2} \} + R_k(\lambda_n) \lambda_n^{2-k/2}
\]

(4)

\[= O \left( \int_0^{\lambda_n} x dx \right) + O(\lambda_n^2)
\]

\[= O(\lambda_n^2).
\]

Hence, from (3), we obtain,

\[| J_1 | = O(\lambda_n).
\]

(5)

Again,

\[
| J_2 | = \left| \sum_{s=n+1}^{\infty} C_s \exp (- \lambda_s t) \right|
\]

\[\leq \lambda_n^{-k/4} \sum_{s=n+1}^{\infty} \lambda_s^{k/4} | C_s \exp (- \lambda_s t) | \]

(6)

\[\leq \lambda_n^{-k/4} \left[ \sum (\nu_1^2 + \cdots + \nu_k^2)^{k/2} | a_{\nu_1 \cdots \nu_k} |^2 \\
\times \sum \exp \{ - 2(\nu_1^2 + \cdots + \nu_k^2) t \} \right]^{1/2}
\]

\[\leq \epsilon_n^{1/2} (\lambda_n)^{-k/4}
\]

(in the last two sums \( \nu_1^2 + \cdots + \nu_k^2 \) runs from \( \lambda_n+1 \) to \( \infty \), where

\[\sum (\nu_1^2 + \cdots + \nu_k^2)^{k/2} | a_{\nu_1 \cdots \nu_k} |^2 = \epsilon_n
\]

(\( \nu_1^2 + \cdots + \nu_k^2 \) runs from \( \lambda_n+1 \) to \( \infty \)), and \( \epsilon_n \to 0 \) as \( n \to \infty \), since

\[\sum e^{-2t^2} = O(t^{-1/2}) \text{ as } t \to 0.
\]

Thus, we have, from (5) and (6),

\[\]
\( S_R(x) - \phi(x, t) = O(\lambda_n) + o(\varepsilon_n^{1/2} (\lambda_n)^{-k/4}) \).

If \( t \) is so chosen that \( t\lambda_n = \delta_n = \varepsilon_n^{1/k} \), then,

\[
S_R(x) - \phi(x, t) = O(\delta_n) + O(\varepsilon_n^{1/2} \delta_n^{-k/4}) = o(1), \quad \text{as } n \to \infty.
\]

**Proof of Theorem II.**

\[
\sum |C_n(x)| \leq \sum |a_{n_1} \cdots a_{n_k}|
\]

\[
\leq \left\{ \sum (v_1^2 + \cdots + v_k^2)^{k/2+\varepsilon} |a_{n_1} \cdots a_{n_k}|^{2} \right\}^{1/2}
\times \left\{ \sum (v_1^2 + \cdots + v_k^2)^{-2(k-1)-\varepsilon} \right\}^{1/2}
\]

\[
= O(1) \sum (\lambda_k \lambda_n^{-k/2-\varepsilon})^{1/2}
\]

\[
= O \left( \left( \int_{-\infty}^{\infty} R_k(x) x^{-k/2-1-\varepsilon} \, dx \right)^{1/2} \right)
\]

\[
= O \left( \left( \int_{-\infty}^{\infty} x^{-1-\varepsilon} \, dx \right)^{1/2} \right) < \infty.
\]

On using Hölder’s inequality instead of Schwarz’s in (3) and (6), we can easily generalize Theorem I as follows:

*If \( \sum (v_1^2 + \cdots + v_k^2)^{(p-1)/2} |a_{n_1} \cdots a_{n_k}|^p < \infty \), where \( 1 < p \leq 2 \), then \( \sum C_n \) converges at every point of continuity of \( f(x) \).*

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