GENERALIZATIONS OF TWO THEOREMS OF JANISZEWSKI. II

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The purpose of this note is to strengthen Theorems 5 and 6 of [1][1] and to make corrections regarding assumptions of compactness in that paper. The following theorems hold in the plane.

THEOREM 1. *If neither of the domains D_1, D_2 separates the point A from the point B, the boundary of D_1 is compact and the common part of D_2 and each component of D_1 is connected or does not exist, then D_1+D_2 does not separate A from B.*

PROOF. Assume that D_1+D_2 separates A from B. Considering there to be a point P at infinity, we find that D_1+D_2+P contains a simple closed curve J separating A from B. Let d_2 be a component of D_2 intersecting J. We find [1, Theorem 4] that $J-J\cdot d_2$ contains a continuum M cutting A from B in the complement of d_2 and such that any open arc of J containing M separates A from B in the complement of d_2. Let d_1 be a component of D_1 covering a point of M on the boundary of d_2. Now d_1 covers M or else it would intersect two components of D_2. But by Theorem 5 of [1], d_1+d_2 does not separate A from B.

Instead of assuming that the boundary of D_1 is compact, we could assume that the part of D_1 in the complement of D_2 is compact.

THEOREM 2. *If neither of the domains D_1, D_2 cuts the point A from the point B, the boundary of D_1 is compact and the common part of D_2 and each component of D_1 is connected or does not exist, then D_1+D_2 does not cut A from B.*

PROOF. Let C_i $(i=1, 2)$ be the component of the complement of D_i containing $A+B$, let D_i' be the complement of C_i and let D_2'' be the sum of all components of D_2' that are not covered by D_1'. Neither D_1' nor D_2'' separates the plane. The boundary of D_1' is a subset of the boundary of D_1 and is therefore compact. If d' is a component of D_1', we shall show that $d'\cdot D_2''$ is connected or does not exist. It will follow from Theorem 1 that $D_1'+D_2''$ does not separate the plane. Hence, its complement is a continuum containing $A+B$ and its subset D_1+D_2 does not cut A from B.

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[1] Number in brackets refers to the reference cited at the end of the paper.

Assume that $d' \cdot D_2''$ contains two components c_1 and c_2. There exist an arc in d' from a point of c_1 to a point of c_2 and a simple closed curve J in D_1 separating this arc from the boundary of d'. Let d be the component of D_1 containing J and let P_iQ_i be an arc in J irreducible from a point P_i of $J \cdot c_i$ to the boundary of c_i. Since Q_i is a point of C_2, P_iQ_i must contain a point of D_2. Then both c_1 and c_2 contain points of $d \cdot D_2$. But it is contrary to a hypothesis of this theorem that $d \cdot D_2$ not be connected. Hence, $d' \cdot D_2''$ is connected or does not exist.

Example. Theorem 2 would not be true if instead of assuming that the boundary of D_1 is compact, we assume that the part of D_1 in the complement of D_2 is compact. Let $D_1 \cdot D_2$ be the set of all points having positive ordinates less than 1 other than those on the lines joining $(1, 1/n)$ to $(n, 1/n)$, $(n, 1/n)$ to $(1, 1)$, $(-1, 1/n)$ to $(-n, 1/n)$ and $(-n, 1/n)$ to $(-1, 1)$ for $n=2, 3, \cdots$; let D_i $(i=1, 2)$ be the sum of $D_1 \cdot D_2$ and the interior of a unit circle with center at $([-1]^i, 1)$. Neither D_1 nor D_2 cuts $(0, 0)$ from $(0, 1)$ but their sum does.

THEOREM 3. *Suppose that neither of the sets H, K cuts the point A from the point B, that the boundary of H is compact, that the junction of H and K is equal to $H \cdot K$ and that H is the sum of a collection of mutually exclusive sets no one of which contains either a limit point of the sum of the others or two components of $H \cdot K$. Then $H+K$ does not cut A from B.*

PROOF. We note that H is contained by a domain D, no component of which contains two components of $H \cdot K$. Let C_H and C_K be two continua in the complements of H and K respectively such that each contains $A+B$. Let D_0 be a subdomain of D with a compact boundary such that D_0 contains $H \cdot K$ but no point of $C_H + C_K$ and each component of D_0 contains a point of $H \cdot K$. There exist domains D_1 and D_2 such that D_1 is a subset of D having a compact boundary and containing $H - H \cdot D_0$ but no point of C_H, D_2 contains $K - K \cdot D_0$ but no point of C_K and $D_1 \cdot D_2$ is a subset of D_0. Considering $D_0 + D_1$ and $D_0 + D_2$ as the domains of Theorem 2, we find that $D_0 + D_1 + D_2$ does not cut A from B. Hence, its subset $H+K$ does not.

THEOREM 4. *If H is a compact closed set cutting the point A from the point B in the complement of the connected set K, then H contains a subset H' irreducible with respect to being a closed set cutting A from B in the complement of K. If K is compact, H' is a continuum that is not separated by any subset of the closure of K.*

The proof is as given in Theorem 7 of [1]. If K is not compact, H' need not be a continuum as is shown by the following example.

Let H be the sum of the points (1, 1) and $(-1, 1)$; let K be the common part of domains D_1 and D_2 described in the example in Theorem 2; let A and B be the points (0, 0) and (0, 1).

Corrections to [1]. The example given in Theorem 2 shows that 6, 7, 10 should have been omitted from the third footnote of [1]. As pointed out in Theorem 4, it is necessary to suppose that K is compact in Theorem 7 of [1]. Accordingly, D must be assumed compact in the fourth footnote of [1].

Reference

1. R. H. Bing, *Generalizations of two theorems of Janiszewski*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 954–960.

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