A NOTE ON POINTWISE NONWANDERING TRANSFORMATIONS

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Let X be a T_1-space and let f be a continuous transformation of X into X In the terminology of G. D. Birkhoff [1, p. 191],[1] a point x of X is said to be *nonwandering* under f provided that to each neighborhood U of x there correspond infinitely many positive integers n for which $U \cap f^n(U) \neq \emptyset$; also, the transformation f is said to be *pointwise nonwandering* provided that each point of X is nonwandering under f.

THEOREM 1. *If f is pointwise nonwandering, then so also is f^k for every positive integer k.*

PROOF. (We make use of a technique employed by Erdös and Stone [2, pp. 126–127].) Suppose k is a positive integer, $x_0 \in X$, and U_0 is a neighborhood (= open neighborhood) of x_0. Let n_1 be a positive integer for which $U_0 \cap f^{n_1}(U_0) \neq \emptyset$. Choose $x_1 \in U_0$ so that $f^{n_1}(x_1) \in U_0$ and a neighborhood U_1 of x_1 so that $U_1 \subset U_0$ and $f^{n_1}(U_1) \subset U_0$. Let n_2 be a positive integer for which $U_1 \cap f^{n_2}(U_1) \neq \emptyset$. Choose $x_2 \in U_1$ so that $f^{n_2}(x_2) \in U_1$ and a neighborhood U_2 of x_2 so that $U_2 \subset U_1$ and $f^{n_2}(U_2) \subset U_1$. Continuing this process, we obtain a sequence $\{n_i\}$ of positive integers and a sequence $\{U_i\}$ of neighborhoods so that $U_i \subset U_{i-1}$ and $f^{n_i}(U_i) \subset U_{i-1}$ $(i=1, 2, \cdots)$. Let r_i denote the integer for which $1 \leqq r_i \leqq k$ and $n_i \equiv r_i \bmod k$. Infinitely many of the r_i are equal to some integer, say r. We may suppose $r_i = r$, $U_i \subset U_{i-1}$ and $f^{n_i}(U_i) \subset U_{i-1}$ $(i=1, 2, \cdots)$. Choose an arbitrary positive integer p. Define $n = \sum_{i=1}^{pk} n_i$. Now $n \equiv 0 \bmod k$. Choose $x \in U_{pk}$. Clearly, $x \in U_0$ and $f^n(x) \in U_0$. Hence, $U_0 \cap f^n(U_0) \neq \emptyset$. Since $n \geqq p$, the proof is completed.

LEMMA 1. *If $f(X) = X$ is a homeomorphism, if A and B are closed connected sets for which $A \cup B = X$, $A \cap B = x \in X$ and $A \cap f(A) \neq \emptyset \neq B \cap f(B)$, and if x is nonwandering, then x is fixed.*

PROOF. Assume x is not fixed. We may suppose that $f(x) \in B$. Now $x \notin f^{-1}(A)$ for in the contrary case $f(x) \in A \cap B = x$. The set $f(A)$ is connected and intersects both A and B. Hence, $x \in f(A)$. There exists a neighborhood U of x such that $U \cap f^{-1}(A) = \emptyset$ and such that

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$U \subset f(A)$ for if the latter condition could not be satisfied, $x \in f(B)$ and $f(x) = f(A \cap B) = f(A) \cap f(B) \ni x$. Now $A \subset f(A)$ since otherwise $A \cap f(B) \neq \emptyset \neq B \cap f(B)$ whence $x \in f(B)$ and as before $f(x) = x$. Thus $A \subset f^n(A)$ for each integer $n \geqq 1$. We conclude that for each integer $n \geqq 1, f^{n+1}(U) \cap f^n(A) = \emptyset$ and, since also $U \subset f(A) \subset f^n(A)$, $U \cap f^{n+1}(U) = \emptyset$. This contradicts the hypothesis that x is nonwandering.

THEOREM 2. *If X is connected and if $f(X) = X$ is a pointwise nonwandering homeomorphism, then each cut point of X is periodic.*

PROOF. Let x be a cut point of X. There exist closed connected sets A and B such that $A \cup B = X$, $A \cap B = x$ and $A \neq x \neq B$. The proof is split into two exhaustive cases. Case I: Either $A \cap f^i(A) \neq \emptyset$ $(i = 1, 2, \cdots)$, or $B \cap f^i(B) \neq \emptyset$ $(i = 1, 2, \cdots)$. Case II: There exist positive integers m and n such that $A \cap f^m(A) = \emptyset$ and $B \cap f^n(B) = \emptyset$.

Suppose Case I occurs. We may assume that $A \cap f^i(A) \neq \emptyset$ $(i = 1, 2, \cdots)$. Since $B - x$ is open, there exists a positive integer k such that $B \cap f^k(B) \neq \emptyset$. By Theorem 1, x is nonwandering under f^k and by Lemma 1, x is therefore fixed under f^k.

Suppose Case II occurs. Now $B \supset f^m(A)$ and $A \supset f^n(B)$. Hence $f^n(B) \supset f^{m+n}(A)$ and $f^m(A) \supset f^{m+n}(B)$. It follows that $A \supset f^{m+n}(A)$ and $B \supset f^{m+n}(B)$. Thus, $f^{m+n}(x) = f^{m+n}(A \cap B) = f^{m+n}(A) \cap f^{m+n}(B) \subset A \cap B = x$. (Actually Case II can never occur.)

COROLLARY 1. *If X is a compact connected semi-locally connected metric space and if $f(X) = X$ is a pointwise nonwandering homeomorphism, then every cyclic element of X which is not an end point is periodic.*

A theorem [3, p. 242] of Schweigert's shows that Corollary 1 permits a weakening of hypothesis from pointwise recurrence (= "pointwise almost periodicity" in the sense of Ayres) to pointwise nonwandering in certain theorems of Ayres and Whyburn on the behavior of cyclic elements under a homeomorphism. The reader is referred to Schweigert's paper [3] for complete references to the work of Ayres and Whyburn.

BIBLIOGRAPHY

1. G. D. Birkhoff, *Dynamical systems*, Amer. Math. Soc. Colloquium Publications, vol. 9, New York, 1927.

2. P. Erdös and A. H. Stone, *Some remarks on almost periodic transformations*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 126–130.

3. G. E. Schweigert, *Fixed elements and periodic types for homeomorphisms on s. l. c. continua*, Amer. J. Math. vol.66 (1944) pp. 229–244.

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