BOOK REVIEW

Jacobian elliptic functions. By E. H. Neville. Oxford University Press, 1944. 16+331 pp. $7.50.

In the development of the theory of elliptic functions it is shown at an early stage that, as far as singularities are concerned, the simplest elliptic functions other than constants are those of order 2. This naturally leads to the classification of elliptic functions of order 2 into those with one double pole (of zero residue) and those with two simple poles in the parallelogram of periods. The Weierstrassian theory of doubly periodic functions, the theory which is still frequently included in a first course of complex variables, starts with a function $\wp(z)$ of the first kind which has the double pole at the origin. By using Liouville's general theorems, elliptic functions with singularities, arbitrary within the permissible limits, are constructed.

The Jacobian theory starts out basically with functions of the second kind, and it is Professor Neville's merit to lay particular stress on this purely function-theoretic classification. Historically, the Jacobian functions arose in connection with the inversion of the Legendre integral

$$u = \int_0^\phi (1 - k^2 \sin^2 \theta)^{-1/2} d\theta$$

as $\operatorname{sn} u = \sin \phi$, $\operatorname{cn} u = \cos \phi$, $\operatorname{dn} u = d\phi/du$. While this manner of introduction of the Jacobian functions is understandable enough in view of the fact that the study of elliptic functions was prompted by the occurrence in physical and geometrical problems of certain integrals which could not be evaluated by elementary functions, it certainly lends a character of arbitrariness to the selection of the standard functions of the theory. The other procedure followed in the past has been to postpone the theory of the functions sn u, cn u, dn u until the study of theta functions. The function sn u is then introduced by means of the relation

$$\operatorname{sn} u = \frac{\theta_3}{\theta_2} \cdot \frac{\theta_1(u/\theta_3^2)}{\theta_4(u/\theta_3^2)},$$

which is hardly calculated to inspire a reader with a sense of the importance and necessity of studying Jacobian functions.

In view of this fact, the author's procedure certainly deserves great praise. After a brief résumé of the Weierstrassian theory, he introduces the three primitive functions, as he calls them. Denote the half-periods of the function $\wp(z)$ by ω_f, ω_g, and introduce for purposes of symmetry ω_h, defined by the relation $\omega_f+\omega_g+\omega_h=0$. Setting $\wp(\omega_i)=e_i$, $i=f, g, h$, the three primitive functions fjz, gjz, hjz are defined by means of the equations

$$fjz = (\wp(z) - e_f)^{1/2}, \quad gjz = (\wp(z) - e_g)^{1/2}, \quad hjz = (\wp(z) - e_h)^{1/2},$$

where the radical in each case is chosen so that, as $z \rightarrow 0$,

$$zfjz \rightarrow 1, \qquad zgjz \rightarrow 1, \qquad zhjz \rightarrow 1.$$

Due to the fact that the radicand in each case has all its zeros of order 2, the above definition yields single-valued functions. Except for an unessential, although significant, change of notation, these functions were also considered by many other authors, such as Jordan and Tannery and Molk. The functions fjz, gjz, hjz are of the second kind, with simple poles at the lattice points and simple zeros at the points congruent to ω_f, ω_g, ω_h, respectively. The periods of a function such as fjz are now $2\omega_f$, $4\omega_g$, $4\omega_h$, and ω_f, ω_g, ω_h are now referred to as quarter-periods. To secure a comprehensive notation, ω_j is introduced as an alternative symbol for the origin, and nine new functions are adjoined to the former three, so as to form a set of twelve functions, each of which has simple zeros congruent with one of the four points ω_j, ω_f, ω_g, ω_h, and simple poles congruent with another of these points. The function which has a zero at ω_p and a pole at ω_q is denoted by pqz, where p and q are distinct and may assume any of the values j, f, g, h. These functions are defined by means of expressions such as

$$jfz = fj(z - \omega_f), \qquad hfz = gj(z - \omega_f), \qquad gfz = hj(z - \omega_f),$$

and so forth, together with their reciprocals. The author calls this set of functions the twelve elementary functions, and the notation is a great aid to memory in bringing directly into evidence the functional structure.

After a discussion of the simpler properties of the elementary functions and their addition theorems, the author embarks on a solution of the difficult inversion problem, which appealed to this reviewer as the most original and interesting part of the book. As is well known, it is easy to show that the relation $\zeta = \wp(z)$ is equivalent to the equation

$$(1) \qquad z = \int_{\zeta}^{\infty} (4t^3 - g_2t - g_3)^{-1/2}dt,$$

where the invariants g_2 and g_3 are expressible in terms of periods of the function. The converse problem of inverting an integral of the form (1) by means of a function $\wp(z)$, taken for suitable periods, the so-called problem of inversion, is far more difficult. This problem is frequently solved by making use of the theory of theta functions. Exactly the same type of problem arises in the study of the twelve elementary elliptic functions. The author examines in detail the elliptic integral

$$z = I(w) = \int_w^\infty (R(w))^{-1/2} dw,$$

where $R(w) = (w^2 - b^2)(w^2 - c^2)$, and the radical is chosen so as to be asymptotic to w^2 as $w \to \infty$ along the path of integration. The resulting discussion is extremely enlightening and shows clearly the important role that topology plays in the problem, a fact which is completely obscured when the problem is solved by means of theta functions. As the author shows, it is relatively easy to establish that the inverse function $w(z)$ is single-valued, analytic, and has no poles other than simple ones. On the other hand, the possibility of the analytic continuation of $w(z)$ over the whole finite plane, which the author picturesquely describes as the ubiquity of the function, is considerably more difficult. To put it a little differently, it has to be shown that, to every complex value z_0, there exists a point w_0 and a path from w_0 to infinity such that

$$z_0 = \int_{w_0}^\infty (R(w))^{-1/2} dw.$$

The author gives two proofs of this: the first, his own, which makes use of set-theoretic considerations, and the second, due to Goursat, which, while shorter, depends on somewhat elaborate algebraic manipulations. In this discussion of the inversion problem the reviewer can not entirely agree with the author in completely ignoring Riemann surfaces. At least as far as the American student is concerned, the notion of Riemann surfaces is sufficiently well known from a first course in complex variables to be used in the relatively simple situations that the inversion problem requires, and would add to the geometric understanding of the problem. In general, the geometric aspects of the theory of elliptic functions, such as the problems of conformal mapping associated with the Jacobian functions, have been allotted an unduly minor place in the book, being confined for the most part to problems and casual references.

The second half of the book is devoted to the study of the Jacobian functions proper. This merely requires a suitable specialization of the preceding theory. Here the author introduces the notion of a Jacobian lattice. A lattice is called Jacobian if $gj\omega_f = 1$. It follows that there is one, and only one, Jacobian lattice similar to a given lattice. The Jacobian elliptic functions are functions with simple poles constructed on a Jacobian lattice. Since this method of construction determines the functions to within a multiplicative constant, the functions are multiplied by such constants as will make the leading coefficient at the origin in each case equal to unity. The previous notation, pqz, is now modified so as to give the classical notation. This is done by writing the quarter-periods ω_f, ω_g, ω_h as K_c, K_n, K_d, respectively, and using K_s, as ω_j before, to denote the origin. We then obtain the classical functions sn u, cn u, dn u as well as the nine Glaisher functions.

Much of the following discussion of the Jacobian functions is to a large extent a specialization of the earlier developments. The elliptic integrals which represent the inversion of the standard Jacobian and Glaisher functions are exhibited and the addition theorems are established. An elegant discussion of the Jacobi and Landen transformations follows. A chapter is devoted to integrals of functions of the type pq^2u, another chapter to the dependence of the functions on the parameter, containing some little known results of Hermite. There is a brief introduction to the theta functions, and the book concludes with the application of the preceding theory to the computation of real integrals and the determination of some conformal maps.

In the introduction the author states that his aim in writing the book was the restoration of the study of Jacobian functions to the elementary curriculum. The reviewer feels sceptical, for various reasons, of the possibility of such a reform in this country. Aside from this issue, however, the book is an excellent one and constitutes a valuable and important addition to the large and distinguished literature on the subject. The style of the author is vivid and picturesque, and the relatively few misprints cause no real difficulty in reading. The reviewer can heartily recommend it to any one with some background in complex variables who wishes to learn the Jacobian theory.

W. SEIDEL