

## ALMOST PERIODICITY, EQUI-CONTINUITY AND TOTAL BOUNDEDNESS

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Let  $X$  be a uniform space; that is to say, let  $X$  be a space provided with a system of indexed neighborhoods  $U_\alpha(x)$  ( $x \in X$ ,  $\alpha = \text{index}$ ), subject to the conditions (A. Weil): (1) If  $x \in X$  and if  $\alpha$  is an index, then  $x \in U_\alpha(x)$ ; (2) If  $\alpha$  and  $\beta$  are indices, then there exists an index  $\gamma$  such that  $x \in X$  implies  $U_\gamma(x) \subset U_\alpha(x) \cap U_\beta(x)$ ; (3) If  $\alpha$  is an index, then there exists an index  $\beta$  such that  $x, y, z \in X$  with  $x, y \in U_\beta(z)$  implies  $x \in U_\alpha(y)$ . Let  $T$  be a topological group with identity  $\sigma$  and let  $f$  be a transformation of  $X \times T$  into  $X$ . We agree to write  $f^t(x)$  or  $f_x(t)$  in place of  $f(x, t)$  ( $x \in X$ ,  $t \in T$ ), whenever we wish. Furthermore, let  $f$  define a transformation group; that is to say, suppose  $f^\sigma(x) = x$  and  $f^s f^t(x) = f^{ts}(x)$  ( $x \in X$ ;  $t, s \in T$ ). We impose continuity conditions on  $f$  as the needs arise.

A subset  $E$  of  $T$  is said to be *relatively dense* provided there exists a compact set  $A$  in  $T$  such that each left translate of  $A$  intersects  $E$ . A point  $x$  of  $X$  is called *almost periodic* provided that if  $U$  is a neighborhood of  $x$ , then there exists a relatively dense set  $E$  in  $T$  for which  $f(x, E) \subset U$ . We observe that the notion of almost periodic point depends on the topology in  $T$ , the strongest type of almost periodicity occurring when  $T$  is provided with the discrete topology. It is easily proved that a set  $E$  in  $T$  is relatively dense if and only if there exists a compact set  $B$  in  $T$  such that  $T = EB$ . The set  $f(x, T)$  is called the *orbit* of the point  $x$ .

**THEOREM 1.** *If the family  $[f^t | t \in T]$  is equi-continuous at  $x$ , if  $f_x$  is continuous on  $T$ , and if  $x$  is almost periodic, then the orbit of  $x$  is totally bounded. Conversely, if the family  $[f^t | t \in T]$  is equi-uniformly continuous and if the orbit of  $x$  is totally bounded, then  $x$  is almost periodic.*

**PROOF.** Suppose the hypotheses of the first statement hold. Let  $\alpha$  be an index. There exists an index  $\beta$  such that the  $\beta$ -neighborhood of each compact set in  $X$  is contained in the union of finitely many  $\alpha$ -neighborhoods. By hypothesis we can find an index  $\gamma$  such that  $f^t(U_\gamma(x)) \subset U_\beta(f^t(x))$  ( $t \in T$ ). There are sets  $E$  and  $A$  in  $T$  such that  $T = EA$ ,  $A$  is compact, and  $f(x, E) \subset U_\gamma(x)$ . Hence,  $f(x, T) \subset f(U_\gamma(x), A) \subset U_\beta(f(x, A))$ . Since  $f(x, A)$  is compact,  $U_\beta(f(x, A))$  is contained in

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Presented to the Society, April 27, 1946; received by the editors March 11, 1946.

the union of finitely many  $\alpha$ -neighborhoods. Thus the orbit of  $x$  is totally bounded.

Suppose the hypotheses of the second statement hold. Let  $\alpha$  be an index. By hypothesis there exists an index  $\beta$  such that  $f^t(U_\beta(y)) \subset U_\alpha(f^t(y))$  ( $y \in X, t \in T$ ). Choose finitely many elements  $t_1, \dots, t_n$  of  $T$  so that  $f(x, T) \subset \bigcup_{i=1}^n V_i$ , where  $V_i = U_\beta(f(x, t_i))$ . If  $t \in T$ , then for some  $i$ ,  $f(x, t) \in V_i$  whence  $f(x, tt_i^{-1}) \in U_\alpha(x)$ . Hence,  $x$  is almost periodic.

**COROLLARY 1.** *If the family  $[f^t | t \in T]$  is equi-uniformly continuous, if  $f_x$  is continuous on  $T$ , and if  $x$  is almost periodic, then  $x$  is almost periodic with respect to the discrete topology in  $T$ .*

**COROLLARY 2 (BOCHNER'S THEOREM).** *If  $x(\tau)$ ,  $-\infty < \tau < +\infty$ , is a complex-valued continuous function, then  $x(\tau)$  is an almost periodic function in the sense of Bohr if and only if each sequence of translates of  $x(\tau)$  contains a uniformly convergent subsequence.*

**PROOF.** Define  $Y$  to be the collection of all complex-valued continuous functions  $y(\tau)$ ,  $-\infty < \tau < +\infty$ , and define indexed neighborhoods in  $Y$  so that  $y \in U_n(y_0)$  if and only if  $|y(\tau) - y_0(\tau)| < 1/n$  ( $-\infty < \tau < +\infty$ ) where the index  $n$  is a positive integer. Construct a transformation group  $g$  in  $Y$  by translation of the functions in  $Y$ , taking  $T$  to be the additive group of reals with its natural topology. Now  $x$  is an almost periodic point if and only if  $x(\tau)$  is an almost periodic function. Clearly, the family  $[g^t | t \in T]$  is equi-uniformly continuous. Also if  $x(\tau)$  is an almost periodic function, then  $x(\tau)$  is uniformly continuous whence  $g_x$  is continuous on  $T$ . Hence, by Theorem 1,  $x$  is an almost periodic point if and only if the orbit of  $x$  is totally bounded. The conclusion follows.

We point out that A. Weil [2, pp. 130–133]<sup>1</sup> has essentially taken Theorem 1 as the definition of an almost periodic point with respect to a transformation group.

If  $X$  is an arbitrary set, if  $Y$  is a uniform space, and if  $\Phi$  is a nonvacuous collection of functions on  $X$  to  $Y$ , then we consider  $\Phi$  to be a uniform space in the following manner: If  $\alpha$  is an index belonging to  $Y$  and if  $\phi \in \Phi$ , then the  $\alpha$ -neighborhood  $U_\alpha(\phi)$  of  $\phi$  is taken to be the set of all elements  $\psi$  of  $\Phi$  such that  $\psi(x) \in U_\alpha(\phi(x))$  for every element  $x$  of  $X$ .

The following lemma will be recognized as a generalization of Ascoli's theorem and its converse.

**LEMMA 1.** *Let  $X$  and  $Y$  be uniform spaces and let  $\Phi$  be a nonvacuous*

<sup>1</sup> Numbers in brackets refer to the Bibliography at the end of the paper.

collection of functions on  $X$  to  $Y$ . If  $X$  and  $Y$  are totally bounded and if the family  $\Phi$  is equi-uniformly continuous, then the space  $\Phi$  is totally bounded. Conversely, if the individual functions in  $\Phi$  are uniformly continuous and if the space  $\Phi$  is totally bounded, then the family  $\Phi$  is equi-uniformly continuous.

PROOF. Suppose  $X$  and  $Y$  are totally bounded and  $\Phi$  is equi-uniformly continuous. Let  $\alpha$  be an index belonging to  $\Phi$  and, hence, to  $Y$ . Choose an index  $\beta$  belonging to  $Y$  so that  $a, b, c \in Y$  with  $a, b \in U_\beta(c)$  implies  $a \in U_\alpha(b)$ . We can find an index  $\gamma$  belonging to  $Y$  and finitely many points  $y_1, \dots, y_n$  of  $Y$  so that  $y \in Y$  implies  $U_\gamma(y) \subset U_\beta(y_j)$  for some integer  $j, 1 \leq j \leq n$ . There exists an index  $\delta$  belonging to  $X$  such that  $\phi(U_\delta(x)) \subset U_\gamma(\phi(x)), (\phi \in \Phi, x \in X)$ . Choose finitely many points  $x_1, \dots, x_m$  of  $X$  so that  $X = \bigcup_{i=1}^m U_\delta(x_i)$ . Hence if  $\phi \in \Phi$ , then to each integer  $i, 1 \leq i \leq m$ , there corresponds at least one integer  $j, 1 \leq j \leq n$ , such that  $\phi(U_\delta(x_i)) \subset U_\beta(y_j)$ . Letting  $I, J$  denote the first  $m, n$  positive integers, each element  $\phi$  of  $\Phi$  defines in the obvious manner a class  $C(\phi)$  of transformations of  $I$  into  $J$ . Choose finitely many elements  $\phi_1, \dots, \phi_r$  of  $\Phi$  so that the class  $\bigcup_{k=1}^r C(\phi_k)$  is maximal. It follows that  $\Phi = \bigcup_{k=1}^r U_\alpha(\phi_k)$ . Hence,  $\Phi$  is totally bounded.

Suppose now that the functions in  $\Phi$  are uniformly continuous and  $\Phi$  is totally bounded. Let  $\alpha$  be an index belonging to  $Y$ . There exists an index  $\gamma$  belonging to  $Y$  so that  $a \in U_\gamma(c), b \in U_\gamma(d)$  and  $c \in U_\gamma(d)$  implies  $a \in U_\alpha(b)$ . Choose finitely many elements  $\phi_1, \dots, \phi_n$  of  $\Phi$  so that  $\Phi = \bigcup_{i=1}^n U_\gamma(\phi_i)$ . Select indices  $\beta(i) (i=1, \dots, n)$  belonging to  $X$  which have the property that  $\phi_i(U_{\beta(i)}(x)) \subset U_\gamma(\phi_i(x)) (x \in X; i=1, \dots, n)$ . There exists an index  $\beta$  belonging to  $X$  for which  $U_\beta(x) \subset \bigcap_{i=1}^n U_{\beta(i)}(x) (x \in X)$ . We conclude that  $\phi(U_\beta(x)) \subset U_\alpha(\phi(x)) (\phi \in \Phi, x \in X)$ . Hence,  $\Phi$  is equi-uniformly continuous and the proof is completed.

We say that the transformation group  $f$  is *almost periodic* provided that if  $\alpha$  is an index, then there exists a relatively dense set  $E$  in  $T$  for which  $x \in X$  implies  $f(x, E) \subset U_\alpha(x)$ . It may be verified that in order for  $f$  to be almost periodic it is both necessary and sufficient that if  $\alpha$  is an index, then there exists a compact set  $A$  in  $T$  such that to each element  $t$  in  $T$  there corresponds an element  $s$  in  $A$  for which  $f^t(x) \in U_\alpha(f^s(x)) (x \in X)$ . If  $T$  has the discrete topology, this characterization reduces essentially to Montgomery's definition [1, p. 323] of an almost periodic transformation group.

In the following theorem we denote  $[f^t | t \in T]$  by  $G$  and, somewhat improperly, call  $G$  the transformation group.

**THEOREM 2.** *If  $X$  is compact and if  $f$  is continuous on  $X \times T$ , then*

*the following statements are pairwise equivalent: (1) The transformation group  $G$  is almost periodic; (2) The family  $G$  is equi-uniformly continuous; (3) The space  $G$  is totally bounded.*

PROOF. By Lemma 1, (2) is equivalent to (3).

Suppose (1) is satisfied. Let  $\alpha$  be an index. There exists an index  $\gamma$  such that  $a \in U_\gamma(c)$ ,  $b \in U_\gamma(d)$  and  $c \in U_\gamma(d)$  implies  $a \in U_\alpha(b)$ . It is possible to find a compact set  $A$  in  $T$  such that if  $t \in T$ , then  $f^t(x) \in U_\gamma(f^s(x))$  ( $x \in X$ ), for some element  $s$  in  $A$ . Since  $f$  is uniformly continuous on  $X \times A$ , we can choose an index  $\beta$  for which  $f^s(U_\beta(x)) \subset U_\gamma(f^s(x))$  ( $x \in X$ ,  $s \in A$ ). It follows that  $f^t(U_\beta(x)) \subset U_\alpha(f^t(x))$  ( $x \in X$ ,  $t \in T$ ). Hence, (2) is satisfied.

Suppose (2) is satisfied. Let  $\alpha$  be an index. There exists an index  $\beta$  such that  $f^t(U_\beta(x)) \subset U_\alpha(f^t(x))$  ( $x \in X$ ,  $t \in T$ ). Since  $G$  is totally bounded, we can select finitely many elements  $t_1, \dots, t_n$  in  $T$  so that  $G = \bigcup_{i=1}^n U_\beta(f^{t_i})$ . Let  $t$  be an element of  $T$ . Then for some  $i$ ,  $f(x, t) \in U_\beta(f(x, t_i))$  ( $x \in X$ ), whence  $f(x, t) \in U_\alpha(x)$  ( $x \in X$ ). Thus (1) is satisfied and the proof is completed.

**COROLLARY 3.** *If  $X$  is compact, if  $f$  is continuous on  $X \times T$ , and if  $f$  is almost periodic, then  $f$  is almost periodic with respect to the discrete topology in  $T$ .*

**COROLLARY 4 (SHARPENED DIRICHLET-KRONECKER THEOREM).** *If  $t, a_1, \dots, a_k$  are nonzero real numbers and if  $\epsilon$  is a positive number, then there exists a relatively dense set  $N$  of integers such that  $n \in N$  implies the existence of integers  $m_1, \dots, m_k$  for which  $|nt - m_i a_i| < \epsilon$  ( $i = 1, \dots, k$ ).*

PROOF. Let  $X_1, \dots, X_k$  denote pairwise disjoint circle boundaries in the plane with circumferences  $|a_1|, \dots, |a_k|$ . Take  $X = \bigcup_{i=1}^k X_i$  with the natural uniformity. Define the uniformity-preserving homeomorphism  $\phi$  of  $X$  onto  $X$  by rotating each circle  $X_i$  through arc length  $t$ . The transformation group generated by the integral powers of  $\phi$  satisfies (2) of Theorem 2 and, hence, is almost periodic. The conclusion follows from the definition of almost periodicity.

#### BIBLIOGRAPHY

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