BOUNDED J-FRACTIONS

H. S. WALL

1. Introduction. A J-fraction

$$(1.1)\qquad \cfrac{1}{b_1 + z - \cfrac{a_1^2}{b_2 + z - \cfrac{a_2^2}{b_3 + z - \ddots}}},$$

in which the coefficients a_p and b_p are complex constants and z is a complex parameter, is said to be *bounded* if there exists a constant M such that

$$(1.2)\qquad |a_p| \leqq M/3, \qquad |b_p| \leqq M/3, \qquad p = 1, 2, 3, \cdots .$$

This condition can be formulated in terms of J-forms in accordance with the following theorem.

THEOREM 1.1. *The J-fraction* (1.1) *is bounded if and only if there exists a constant N such that*

$$(1.3)\qquad \begin{aligned} &\left| \sum_{p=1}^{n} b_p u_p v_p - \sum_{p=1}^{n-1} a_p(u_p v_{p+1} + u_{p+1} v_p) \right| \\ &\qquad \leqq N \left(\sum_{p=1}^{n} |u_p|^2 \cdot \sum_{p=1}^{n} |v_p|^2 \right)^{1/2}, \qquad n = 1, 2, 3, \cdots , \end{aligned}$$

for all values of the variables u_p and v_p, the constant N being independent of the variables and of n.

In fact, if (1.3) holds then we find, on specializing the values of the u_p and v_p, that $|b_p| \leqq N$, $|a_p| \leqq N$, $p=1, 2, 3, \cdots$; and if (1.2) holds then, by Schwarz's inequality, (1.3) holds with $N=M$.

If (1.3) holds, then the J-form $\sum b_p u_p v_p - \sum a_p(u_p v_{p+1} + u_{p+1} v_p)$ is said to be *bounded*, and the least value of N which can be used in that inequality is called the *norm* of the J-form. We shall also call this number the *norm* of the J-fraction. When (1.2) holds then, as pointed out above, (1.3) holds with $N=M$. *Hence the norm of the J-fraction does not exceed the least number M which can be used in* (1.2).

Presented to the Society, April 27, 1946; received by the editors February 28, 1946.

THEOREM 1.2. *If* (1.2) *holds, then the J-fraction converges uniformly for* $|z| \geqq M$.

For if the J-fraction is transformed by an equivalence transformation so that all the partial denominators are equal to unity, then the nth partial numerator is

$$-\frac{a_{n-1}^2}{(b_{n-1}+z)(b_n+z)}.$$

If $|z| \geqq M$, and (1.2) holds, this has modulus not greater than 1/4. Hence it follows by a well known theorem that the J-fraction converges uniformly for $|z| \geqq M$.

In the case of the J-fraction

$$\cfrac{1}{1+z-\cfrac{1}{1+z-\cfrac{1}{1+z-\cdots}}},$$

the least number M which can be used in (1.2) is $M=3$. Hence the J-fraction converges uniformly for $|z| \geqq 3$. It diverges if z is real, positive, and less than 3. On the other hand, for the J-fraction

$$\cfrac{1}{z-\cfrac{(1/4)}{z-\cfrac{(1/4)}{z-\cdots}}},$$

the least value of M which can be used in (1.2) is $M=3/2$. The norm of this J-fraction is $N=1$, and it converges for $|z| \geqq 1$. In fact, it converges if z is not on the real interval $-1<x<+1$.

The principal object of this note is to show that a J-fraction with norm N converges if z is not in a certain convex set contained in the circle $|z|=N$. Moreover, if the partial numerators $a_p{}^2$ are different from zero, the corresponding J-matrix has a unique bounded reciprocal for all z not in this convex set. It was shown by Hellinger and Toeplitz [3][1] that there is a unique bounded reciprocal for $|z|>N$.

[1] Numbers in brackets refer to the Bibliography at the end of the paper.

2. **Convergence of bounded J-fractions.** Let $a=e^{i\theta}$ be a complex number with modulus unity. Then the J-fractions (1.1) and

$$(2.1)\qquad \cfrac{a}{b_1a+Z-\cfrac{(a_1a)^2}{b_2a+Z-\cfrac{(a_2a)^2}{b_3a+Z-\cdots}}}\,, \qquad Z = az,$$

are equivalent in the sense that their nth approximants are identical with one another for $n=1, 2, 3, \cdots$. Also, they obviously have one and the same norm.

Let

$$(2.2)\qquad \alpha_p(\theta) = I(a_pa), \qquad \beta_p(\theta) = I(b_pa), \qquad p = 1, 2, 3, \cdots.$$

Then, if (1.1) is bounded, it follows from (1.3) that there exists a finite constant $Y(\theta)$ such that

$$(2.3)\qquad \sum_{p=1}^{n} [\beta_p(\theta) + Y(\theta)]x_p^2 - 2\sum_{p=1}^{n-1} \alpha_p(\theta)x_px_{p+1} \geqq 0, \quad n = 1, 2, 3, \cdots,$$

for all real values of $x_1, x_2, x_3, \cdots$. If $Y_0(\theta)$ is the least value of $Y(\theta)$ which can be used in (2.3), then we must have

$$| Y_0(\theta) | \leqq N, \qquad 0 \leqq \theta < 2\pi,$$

where N is the norm of the J-fraction.

From (2.3) it follows that if we put $Z=iY(\theta)+\zeta$ in (2.1), then (2.1) is a positive definite J-fraction in the variable ζ. Therefore, if c is a positive constant, the nth approximant of (1.1), which is the same as the nth approximant of (2.1), satisfies the inequality [1]

$$\left| \frac{A_n(z)}{B_n(z)} \right| \leqq \frac{1}{c},$$

provided $I(\zeta) \geqq c$, that is, provided

$$(2.4)\qquad x \sin\theta + y\cos\theta \geqq Y(\theta) + c, \qquad \text{where } z = x + iy;$$

and $B_n(z) \neq 0$ when (2.4) holds. This can be interpreted geometrically as follows. Let K denote the set of all points $z=x+iy$ such that

$$x\sin\theta + y\cos\theta \leqq Y(\theta) \qquad \text{for } 0 \leqq \theta < 2\pi.$$

Then, K is a convex set of which the straight lines $x \sin \theta + y \cos \theta = Y(\theta)$ are the supporting lines; the zeros of all the denominators $B_n(z)$ of the J-fraction (1.1) are in K; the approximants of the J-fraction are uniformly bounded over any domain whose distance from K is positive. We shall let K_0 denote the convex set determined in this way corresponding to the function $Y_0(\theta)$ defined above.

By Theorem 1.2, the J-fraction converges if $|z|$ is sufficiently large. We may then conclude immediately by a theorem of Stieltjes [6] that the following theorem is true.

THEOREM 2.1. *A bounded J-fraction converges uniformly over every bounded closed region whose distance from the convex set K_0 is positive. In particular, the J-fraction converges if $|z| > N$, where N is the norm of the J-fraction.*

We note the following special cases. If the coefficients a_p and b_p are all real, then $Y_0(0) = Y_0(\pi) = 0$, so that K_0 reduces to an interval of the real axis contained in the interval $-N \leqq x \leqq +N$. If the a_p are pure imaginary and the b_p are real and positive, then the set K_0 is contained in the left half-plane, $x = R(z) \leqq 0$.

3. **Bounds for the zeros of a polynomial.** The preceding considerations furnish a method for determining bounds for the zeros of a polynomial. Let $P(z)$ be a polynomial of degree n, $n > 1$, and let $Q(z)$ be any polynomial of degree $n-1$ such that there is a continued fraction expansion of the form

$$\frac{Q(z)}{P(z)} = \cfrac{c}{b_1 + z - \cfrac{a_1^2}{b_2 + z - \ddots - \cfrac{a_{n-1}^2}{b_n + z}}}, \tag{3.1}$$

where $a_p \neq 0$, $p = 1, 2, 3, \cdots, n-1$, and $c \neq 0$. This expansion can be easily obtained by applying the euclidean algorithm for the greatest common divisor to $Q(z)$ and $P(z)$. Let K_0 be the convex set which is associated with this J-fraction in the manner indicated in §2. Then the zeros of $P(z)$ are all contained in K_0.

One may readily obtain a rectangle containing the set K_0. In fact, if we use the notation of §2, such a rectangle is given by

$$y \leqq Y(0), \qquad x \leqq Y(\pi/2),$$
$$y \geqq -Y(\pi), \qquad x \geqq -Y(3\pi/2).$$

This rectangle is obtained by minimizing four *real* quadratic forms.

By way of illustration, let $P(z)=z^3+(2+i)z^2+(3+i)z+(2i+2)$, and take $Q(z)=2z^2+iz+2$. Then,

$$\frac{Q(z)}{P(z)}=\cfrac{2}{(2+i/2)+z-\cfrac{(3i/2)^2}{-i/6+z-\cfrac{(8^{1/2}i/3)^2}{2i/3+z}}}\cdot$$

We require lower bounds $-Y(\theta)$ for the quadratic form

$$(2\sin\theta+(1/2)\cos\theta)x_1^2-(1/6)\cos\theta x_2^2+(2/3)\cos\theta x_3^2 \\ -3\cos\theta x_1x_2-(2^{5/2}/3)\cos\theta x_2x_3,$$

under the condition $x_1^2+x_2^2+x_3^2=1$, and for $\theta=0,\ \pi/2,\ \pi,\ 3\pi/2$. Easily determined lower bounds are given by

$$Y(0)=19/6,\qquad Y(\pi/2)=0,\qquad Y(\pi)=11/3,\qquad Y(3\pi/2)=2.$$

Therefore, the zeros of $P(z)$ are contained in the rectangle

$$y\leqq 19/6,\qquad x\leqq 0,$$
$$y\geqq -11/3,\qquad x\geqq -2.$$

The zeros of $P(z)$ are actually equal to

$$-1-i,\qquad \frac{-1-7^{1/2}i}{2},\qquad \frac{-1+7^{1/2}i}{2}\cdot$$

The size of the rectangle depends upon the choice of the polynomial $Q(z)$. In fact, it is easy to show that the zeros of $Q(z)$ also lie in the convex set K_0. Furthermore, the computational difficulties are less for some choices of $Q(z)$ than they are for other choices. Let

$$P(z)=z^n+(p_1+iq_1)z^{n-1}+(p_2+iq_2)z^{n-2}+\cdots+(p_n+iq_n).$$

Then, if

$$Q(z)=p_1z^{n-1}+iq_2z^{n-2}+p_3z^{n-3}+iq_4z^{n-4}+\cdots,$$

the computation involved in obtaining the J-fraction expansion for $Q(z)/P(z)$ is especially simple. Moreover, from this expansion one can determine immediately the number of zeros of $P(z)$ in each of the half-planes $R(z)<0$ and $R(z)>0$. For details, we refer the reader to a recent paper of Frank [2].

4. The bounded reciprocal of a bounded J-matrix. We suppose that (1.1) is bounded, that $a_p \neq 0$, $p=1, 2, 3, \cdots$, and consider the J-matrix

$$(4.1) \quad J + zI = \begin{Vmatrix} b_1 + z, & -a_1, & 0, & 0, & 0, & \cdots \\ -a_1, & b_1 + z, & -a_2, & 0, & 0, & \cdots \\ 0, & -a_2, & b_2 + z, & -a_3, & 0, & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{Vmatrix}.$$

If the norm of (1.1) is N, and $|z| > N$, then the matrix $J+zI$ has a unique bounded reciprocal which is given by

$$(J + zI)^{-1} = \frac{I}{z} - \frac{J}{z^2} + \frac{J^2}{z^3} - \cdots .$$

This is a matrix whose elements are power series in $1/z$, convergent for $|z| > N$. In particular, the element in the first row and first column is the power series expansion of the J-fraction, and its sum is the value of the J-fraction (Hellinger and Toeplitz [3]).

We can now show that $J+zI$ has a unique bounded reciprocal for any z not in the set K_0 defined in §2. In fact, if we put $Z=iY_0(\theta)+\zeta$ in (2.1), then, as we have seen, the J-fraction is a positive definite J-fraction in the variable ζ. The corresponding J-matrix is

$$(4.2) \qquad aJ + iY_0(\theta)I + \zeta I.$$

Inasmuch as the series $\sum(1/|aa_p|)$ is divergent, the *determinate case* holds for the J-fraction [1] and consequently [7] the matrix (4.2) has a unique bounded reciprocal for $I(\zeta) > 0$. *We therefore conclude immediately that the J-matrix $J+zI$ has a unique bounded reciprocal for any z not in the set K_0 defined in* §2.

5. **Functions represented by J-fractions.** Every infinite subsequence of approximants of a positive definite J-fraction contains an infinite subsequence which converges for $I(z) > 0$ to a function which is analytic and has a negative imaginary part in this domain, and which has the form

$$(5.1) \qquad f(z) = \int_{-\infty}^{+\infty} \frac{d\phi(u)}{z - u},$$

where $\phi(u)$ is bounded and nondecreasing. There are functions which are analytic and have negative imaginary parts for $I(z) > 0$ which are not limits of sequences of approximants of positive J-fractions. In fact, the most general function of this description has the form

(5.2)
$$\int_{-\infty}^{+\infty} \frac{1+zu}{z-u}\,d\phi(u) + a - bz = \int_{-\infty}^{+\infty}\left(\frac{u}{1+u^2} + \frac{1}{z-u}\right)(1+u^2)\,d\phi(u) + a - bz,$$

where a and b are real, $b \geqq 0$, and $\phi(u)$ is bounded and nondecreasing. This can be seen as follows. F. Riesz [5] and Herglotz [4] showed that a function $f(w)$ is analytic and has a positive real part for $|w| < 1$ if and only if it has the form

$$f(w) = \int_0^{2\pi} \frac{e^{it}+w}{e^{it}-w}\,d\sigma(t) + ia,$$

where a is real and $\sigma(t)$ is bounded and nondecreasing. If we multiply this integral by $-i$ and make the substitution

$$w = \frac{1+iz}{1-iz}, \tag{5.3}$$

mapping the unit circle upon the upper half-plane, we obtain after simple transformations

$$\int_0^{2\pi} \frac{1 - z\tan(t/2)}{\tan(t/2) + z}\,d\sigma(t) + a.$$

This can be transformed into (5.2) if we put $u = \tan(t/2)$.

We take this occasion to point out that there exists an identical continued fraction transformation of the integral (5.2). We have the following theorem.

THEOREM 5.1. *A necessary and sufficient condition for a function to be analytic and have a negative imaginary part for $I(z) > 0$ is that it have a continued fraction expansion of the form*

(5.4
$$\cfrac{c}{z - r_0 - \cfrac{g_1(1+z^2)}{z - r_1 - \cfrac{(1-g_1)g_2(1+z^2)}{z - r_2 - \cfrac{(1-g_2)g_3(1+z^2)}{z - r_3 - \ddots}}}},$$

where $c > 0$, $0 < g_p < 1$, $-\infty < r_{p-1} < +\infty$, $p = 1, 2, 3, \cdots$, or a terminat-

ing continued fraction expansion of this form in which the last g_p *which appears may be equal to unity. The continued fraction converges uniformly over every bounded closed region within the half-plane* $I(z)>0$, *and is uniquely determined by the function expanded.*

To prove this, it is only necessary to make the substitution

$$z = \frac{4w}{(1 - w)^2}$$

in the continued fraction (3.14) of [8], multiply the resulting continued fraction by $-i$, and then make the substitution (5.3) above.

BIBLIOGRAPHY

1. J. J. Dennis and H. S. Wall, *The limit-circle case for a positive definite J-fraction*, Duke Math. J. vol. 12 (1945) pp. 255–273.
2. Evelyn Frank, *On the zeros of polynomials with complex coefficients*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 144–157.
3. E. Hellinger and O. Toeplitz, *Zur Einordnung der Kettenbruchtheorie in die Theorie der quadratischen Formen von unendlichvielen Veränderlichen*, Journal für Mathematik vol. 14 (1914) pp. 212–238.
4. G. Herglotz, *Über Potenzreihen mit positivem, reellem Theil im Einheitskreis*, Berichte über die verhandlungen der K. Sächsische Akademie der wissenschaften zu Leipzig vol. 63 (1911) pp. 501–511.
5. F. Riesz, *Sur certains systems singuliérs d'équations intègrales*, Ann. École Norm. (3) vol. 28 (1911) pp. 34–62.
6. T. J. Stieltjes, *Œuvres*, vol. 2, p. 451 ff.
7. H. S. Wall, *Reciprocals of J-matrices*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 680–685.
8. ———, *Continued fraction expansions for functions with positive real parts*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 138–143.

ILLINOIS INSTITUTE OF TECHNOLOGY