A NOTE ON LINEAR HOMOGENEOUS DIOPHANTINE EQUATIONS

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In this paper the coefficients a_{ij} in the equations

$$(1)\qquad a_{i1}x_1 + \cdots + a_{in}x_n = 0 \qquad (i = 1, \cdots, m)$$

are constant rational integers and all letters denote integers. If $m=n-1$ and the rank is $n-1$ then the complete solution in integers is well known. Thus, if E_j is the determinant obtained by deleting the jth column from the matrix of the coefficients, and if $e=(E_1, \cdots, E_n)$, then the solution is

$$(2)\qquad x_j = (-1)^j tE_j/e \qquad (j = 1, \cdots, n),$$

in which t is an arbitrary integer.

E. T. Bell recently conjectured that if $m<n-1$ and if the rank r is m then the solution is similarly obtained from the system formed by (1) and the equations

$$(3)\qquad \xi_{i1}x_1 + \cdots + \xi_{in}x_n = 0 \qquad (i = 1, \cdots, n - m - 1),$$

in which the ξ_{ij} are arbitrary integers. In this paper this conjecture is proved by induction. Since this solution is written down directly from (1) and is fully displayed these results are more usable than those in the literature.[1]

If $r=1$ it can be assumed without limitation that $a_1 \cdots a_n \neq 0$, $(a_1, \cdots, a_n)=1$, and at least one of $x_1, \cdots, x_n$ is not zero. If $n=3$ there are integers $t, y_1, y_2, y_3, d, A_1, A_2, k_1, k_2$ such that

$$(4)\qquad x_1 = ty_1, \qquad x_2 = ty_2, \qquad x_3 = ty_3, \qquad (y_1, y_2, y_3) = 1,$$

$$(5)\qquad a_1 = dA_1, \qquad a_2 = dA_2, \qquad (A_1, A_2) = 1, \qquad k_1A_2 - k_2A_1 = 1.$$

Since $(d, a_3)=1$ there is an integer s such that

$$(6)\qquad y_3 = ds, \qquad A_1y_1 + A_2y_2 + a_3s = 0.$$

Then since $(A_1, A_2)=1$ there is an integer r such that

$$(7)\qquad y_1 - a_3k_2s = A_2r, \qquad y_2 + a_3k_1s = -A_1r.$$

These conditions are also sufficient. Hence the complete solution is

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[1] Th. Skolem, *Diophantische Gleichungen*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 5, no. 4, 1938; D. N. Lehmer, Proc. Nat. Acad. Sci. U.S.A. vol. 4 (1919).

$$(8)\qquad x_1 = t\cdot\frac{1}{d}\begin{vmatrix} dk_2s & -r \\ a_2 & a_3 \end{vmatrix},\quad x_2 = -t\cdot\frac{1}{d}\begin{vmatrix} dk_1s & -r \\ a_1 & a_3 \end{vmatrix},\quad x_3 = t\cdot\frac{1}{d}\begin{vmatrix} dk_1s & dk_2s \\ a_2 & a_3 \end{vmatrix}.$$

Since $(d, a_3)=1$ there are integers p and P such that $-r+pa_3=dP$. Hence d is the greatest common divisor of the determinants in (8). Therefore for the particular values $-dk_1s$, $-dk_2s$, r of ξ_{11}, ξ_{12}, ξ_{13}, (8) is an instance of (2).

Similarly, if $n=4$ and $d_1=(a_1, a_2)$, $d_3=(a_3, a_4)$ then the solution of $a_1x_1+a_2x_2+a_3x_3+a_4x_4=0$ is equivalent to that of $A_1y_1+A_2y_2=-d_3S$ and $A_3y_3+A_4y_4=d_1S$. By the preceding discussion the complete solution for each of these equations is known. The expressions for S are then equated. Thus it is proved that there are integers $k_1, \cdots, k_4$, r_1, r_3, t_1, t_3, s such that the solution is obtained by applying (2) to the matrix

$$(9)\qquad \begin{bmatrix} -r_3A_1 & -r_3A_2 & d_3k_3t_1s & d_3k_4t_1s \\ d_1k_1t_3s & d_1k_2t_3s & -r_1A_3 & -r_1A_4 \\ d_1A_1 & d_1A_2 & d_3A_3 & d_3A_4 \end{bmatrix}.$$

If $n\geqq 5$ the notations $x_j=ty_j$ $(j=1, \cdots, n)$, $(y_1, \cdots, y_n)=1$, $a_j=\delta\alpha_j$ $(j=1, \cdots, n-2)$, $a_j=\zeta A_j$ $(j=n-1, n)$, $(\alpha_1, \cdots, \alpha_{n-2})=1$, $(A_{n-1}, A_n)=1$, $\alpha_j=d_1A_j$ $(j=1, \cdots, n-3)$, $(A_1, \cdots, A_{n-3})=1$, $\alpha_{n-2}=d_3A_{n-3}$, $\zeta=d_3\eta$, $(A_{n-3}, \eta)=1$ are used. The solution of $a_1x_1+\cdots+a_nx_n=0$ is equivalent to that of $A_{n-1}y_{n-1}+A_ny_n=\delta S$ and $\alpha_1y_1+\cdots+\alpha_{n-2}y_{n-2}=-\zeta S$. In particular, if $n=5$ then y_4, y_5, S are obtained as in (8) and y_1, y_2, y_3, S from (9) with A_4 replaced by η. If, to simplify the notation, f_{ij} denotes the element in the ith row and jth column of this new matrix then $x_1, \cdots, x_5$ are obtained by applying (2) to the matrix

$$(10)\qquad \begin{bmatrix} -r_4d_1A_1 & -r_4d_1A_2 & -r_4d_3A_3 & d_3\eta q_4S_4t_4 & d_3\eta q_5S_4t_4 \\ \delta f_{11} & \delta f_{12} & \delta f_{13} & f_{14}A_4 & f_{14}A_5 \\ \delta f_{21} & \delta f_{22} & \delta f_{23} & f_{24}A_4 & f_{24}A_5 \\ \delta d_1A_1 & \delta d_1A_2 & \delta d_3A_3 & d_3\eta A_4 & d_3\eta A_5 \end{bmatrix}.$$

In general, the matrix for $x_1, \cdots, x_n$ is obtained from the matrix for $n-1$ variables in precisely the same way as (10) is obtained from (9).

If $1<r<n-1$ it can be assumed without limitation that the first equation satisfies the conditions $a_1\cdots a_n\neq 0$, $(a_1, \cdots, a_n)=1$,

of the preceding discussion. If E_i denotes $a_{i1}x_1 + \cdots + a_{in}x_n$ $(i=1, \cdots, r)$ then a given solution of $E_1=0$ is obtained by applying (2) to an appropriate matrix whose ith row is $\alpha_{i1}, \cdots, \alpha_{in}$ $(i=1, \cdots, n-2)$ and whose last row is $a_{11}, \cdots, a_{1n}$. By hypothesis these values of $x_1, \cdots, x_n$ also satisfy $E_2=0$. Therefore the determinant obtained by placing the row $a_{21}, \cdots, a_{2n}$ under this matrix is zero. Hence, if A_i is defined as $\alpha_{i1}x_1 + \cdots + \alpha_{in}x_n$ $(i=1, \cdots, n-2)$, then the rank s of the functions $A_1, \cdots, A_{n-2}$, E_1, E_2 is less than n. Therefore there is a set of s functions from this list on which each remaining function is linearly dependent, with integer coefficients. If indeed the subset includes both E_1 and E_2 then the notation can be assigned so that the subset is $A_1, \cdots, A_{s-2}, E_1$, E_2. Then there are integers $d, d_1, \cdots, d_s$ such that $d \neq 0$ and

$$dA_{n-2} = d_1A_1 + \cdots + d_{s-2}A_{s-2} + d_{s-1}E_1 + d_sE_2. \tag{11}$$

Now the result of applying (2) to the original matrix is the same as the result of applying it to the matrix which is obtained by replacing $\alpha_{n-2,j}$ by $d\alpha_{n-2,j}$ $(j=1, \cdots, n)$. By (11) in this new matrix $d\alpha_{n-2,j}$ may be replaced by the sum $d_1\alpha_{1j} + \cdots + d_{s-2}\alpha_{s-2,j} + d_{s-1}a_{1j} + d_sa_{2j}$, and hence by d_sa_{2j}, and hence by a_{2j}. Again, if the subset includes E_2 but not E_1, or if it includes neither E_1 nor E_2, or if it includes E_1 but not E_2, then in a similar way the matrix can be replaced by a matrix having $a_{11}, \cdots, a_{1n}$ as last row and $a_{21}, \cdots, a_{2n}$ as another row. This process can be continued until $a_{i1}, \cdots, a_{in}$ $(i=1, \cdots, r)$ appear.

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