NONLINEAR NETWORKS. I

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The object of this note is to show that a certain system of nonlinear differential equations has a unique asymptotic solution, that is, all solutions approach each other as the independent variable becomes infinite. The interest of these equations is that they describe the vibrations of electrical networks so we shall first discuss the physical origin of the equations.

A linear network is a collection of linear inductors, linear resistors and linear capacitors arbitrarily interconnected. Suppose that such a network has no undamped free vibration. Then a given impressed force may give rise to more than one response but as time goes on the transient vibrations die out and there is a unique relation between impressed force and response. This, of course, is well known. Our main theorem states that if in such a network the linear resistors are replaced by *quasi-linear* resistors then again, after sufficient time has elapsed, there is a unique relation between impressed force and response.

A quasi-linear resistor is a conductor whose differential resistance lies between positive limits. No other nonlinearity besides this type of nonlinear damping is considered. Quasi-linear resistors have extensive practical application.

For example, consider a linear network with one degree of freedom. An inductor of inductance L, a resistor of resistance R and a capacitor of capacitance S^{-1} are connected in series. The current $y(t)$ flowing in this circuit must satisfy the following differential equation

$$L\frac{dy}{dt} + Ry + S\int ydt = e.$$

Here $e(t)$ is the electromotive force impressed in the circuit and may be an arbitrary function of time.

The corresponding nonlinear equation to be studied is obtained by replacing the linear relation Ry by a function $V(y)$ which for all values of y is such that $\delta \leqq V'(y) \leqq \Delta$, where δ and Δ are positive constants.

In the general network with n degrees of freedom a set of n independent circuits (meshes) is chosen. Then any distribution of cur-

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rent in the network may be uniquely specified in terms of the cyclic currents $y_1, y_2, \cdots, y_n$ flowing in these circuits. Let $e_1, e_2, \cdots, e_n$ be the electromotive forces acting in these circuits. It is convenient to introduce the electric charge variables $q_1, q_2, \cdots, q_n$ such that $y_j = dq_j/dt$ and to let p stand for d/dt. The linear network equations may then be written as

$$Lp^2q + Rpq + Sq = e. \tag{1}$$

Here L, R, and S are n-way matrices and q and e are vectors with components q_j and e_j.

A condition which must be satisfied in order that there be no undamped solutions is specified by the following well known lemma.

LEMMA 1. *A necessary condition that all solutions $y = pq$ of the homogeneous equation $Lp^2q + Rpq + Sq = 0$ approach zero as $t \to +\infty$ is that the matrix $(z^2L + zR + S)$ be nonsingular for $\Re z \geqq 0$ and $|z| > 0$.*

PROOF. Otherwise there is a constant vector q_0 and a value of z such that $(z^2L + zR + S)q_0 = 0$, so $q = q_0 e^{zt}$ is a solution. But $pq = zq_0e^{zt}$ does not approach zero if $\Re z \geqq 0$ and $|z| > 0$.

Well known arguments concerning electric and magnetic energy show that the matrices L, R, and S are symmetric semi-definite. For instance if $\{a, b\}$ indicates the bilinear form $\sum_1^n a_j b_j$ and $\|a\|^2 = \{a, a\}$, then $\{y, Ry\}$ is by definition the rate at which energy is dissipated in the resistors. On the other hand let the resistance of the resistors be $r_1, r_2, \cdots, r_m$ and the corresponding currents through them be $i_1, i_2, \cdots, i_m$. (Each i_ν is, of course, the algebraic sum of those currents y_j which are common to r_ν.) The potential drop across the resistor ν is $r_\nu i_\nu$ and the jth component of Ry is defined as the algebraic sum of those drops which are common to the jth circuit. The following identity must hold for physical reasons

$$\{y, Ry\} = \sum_1^m r_\nu i_\nu^2. \tag{2}$$

This shows that R is non-negative definite.

In the nonlinear network the linear expression $r_\nu i_\nu$ is replaced by the function $v_\nu(i_\nu)$ where $\delta \leqq v_\nu' \leqq \Delta$ for positive constants δ and Δ. Let $V(y)$ be the vector function which replaces Ry in equation (1). If y^+ and y^* are two arbitrary vectors we can write $V(y^+) - V(y^*) = V' \cdot (y^+ - y^*)$, where V' is a matrix defined as follows. By the mean value theorem

$$v_\nu(i_\nu^+) - v_\nu(i_\nu^*) = v_\nu'(\theta)(i_\nu^+ - i_\nu^*),$$

where θ lies between i_ν^+ and i_ν^*. Thus the matrix V' is defined by the v_ν' the same way that R is defined by the r_ν.

Without loss of generality we may suppose $\delta \leqq r_\nu \leqq \Delta$ also. Then by identity (2)

$$(\delta/\Delta)\{y, Ry\} \leqq \{y, V'y\} \leqq (\Delta/\delta)\{y, Ry\}.$$

Before proceeding with the uniqueness proof it is desirable to give an abstract definition of this nonlinear transformation so that a theorem can be stated which is independent of physical concepts. Let R be a symmetric semi-definite matrix. Then a continuous vector function $V(y)$ is a *quasi-linear replacement* of R provided $V(y^+) - V(y^*) = V' \cdot (y^+ - y^*)$, where V' is a symmetric matrix which satisfies $k^{-1}\{y, Ry\} \leqq \{y, V'y\} \leqq k\{y, Ry\}$ for a positive constant k independent of the vectors y^+, y^*, and y.

THEOREM. *Let L, R, and S be symmetric semi-definite matrices and suppose that all solutions of the equation $Ld^2x/dt^2 + Rdx/dt + Sx = 0$ are such that $dx/dt \to 0$ as $t \to +\infty$. Let V be a quasi-linear replacement of R and suppose for $t \geqq 0$ the vectors $q(t)$ and $e(t)$ satisfy the equation $Ld^2q/dt^2 + V(dq/dt) + Sq = e$. Then if q^* is any other solution of this equation, $\int_0^\infty \| dq/dt - dq^*/dt \|^2 dt < \infty$. (The first derivatives of x, q, and q^* are assumed to be continuous.)*

PROOF. Let $w = q - q^*$; then by subtraction

$$Lp^2w + V'pw + Sw = 0. \tag{3}$$

This gives the bilinear form

$$\{pw, Lp^2w\} + \{pw, V'pw\} + \{pw, Sw\} = 0,$$

so

$$-p[\{pw, Lpw\} + \{w, Sw\}] = 2\{pw, V'pw\}.$$

By definition of V' the function on the right is continuous so by a standard theorem of the integral calculus

$$-[\{pw, Lpw\} + \{w, Sw\}] + A = 2\int_0^t \{pw, V'pw\} dt.$$

But L and S are non-negative definite so $\int_0^\infty \{pw, V'pw\} dt < A/2$. The integral is absolutely convergent because the integrand is non-negative. (We shall use the letter A to designate various other constants also.)

The following simple properties of a symmetric semi-definite matrix H are needed.

LEMMA 2. *If, for all* x, $\{x, Hx\} \leqq A\|x\|^2$ *then* $\|Hx\|^2 \leqq A\{x, Hx\}$.

LEMMA 3. *The equation* $a = Hx$ *is solvable for* x *if and only if the relation* $Hu = 0$ *implies* $\{a, u\} = 0$.

LEMMA 4. *There is a matrix* P *such that, for any* x, $Hx = HPx$ *and* $\|Px\| \leqq A\|Hx\|$, *where* A *is independent of* x.

PROOF. These lemmas are obvious if H is a diagonal matrix. However, the relations are invariant under a rotation of axes (orthogonal transformation) and it is well known that there is a rotation which reduces H to diagonal form.

For each value of t the matrix V' is non-negative definite and $\{x, V'x\} \leqq A\|x\|^2$, where A is independent of t. Thus by Lemma 2, $\int_0^\infty \|V'pw\|^2 \leqq A\int_0^\infty \{pw, V'pw\}dt < \infty$. Also $\{x, Rx\} \leqq k\{x, V'x\}$; so again by Lemma 2, $\int_0^\infty \|Rpw\|^2 dt < \infty$.

From $\{x, V'x\} \leqq k\{x, Rx\}$ it follows by Lemma 2 that $\|V'x\|^2 \leqq Ak\{x, Rx\}$. This implies $V'x = 0$ if $Rx = 0$. Now by Lemma 3 it follows that if $a = V'b$ there is an x such that $a = Rx$. Otherwise there would be a u such that $Ru = 0$ and $\{a, u\} \neq 0$. But then $V'u = 0$; so $\{a, u\} = 0$, a contradiction.

Let $b - x = f_0$; then $Rb - V'b = Rf_0 = RPf_0$. Let $f = Pf_0$; then by Lemma 4

$$\|f\| \leqq A\|Rf\| \leqq A\|Rb\| + A\|V'b\|.$$

Interpreting b as pw, it follows that $\int_0^\infty \|f\|^2 dt < \infty$, that is, $f(t) \subset L_2(0, \infty)$. Moreover

$$\|f(t_1) - f(t_2)\| \leqq A\|Rpw_1 - Rpw_2\| + A\|V_1' pw_1 - V_2' pw_2\|.$$

Since pw and V' are continuous it follows that $f(t)$ is continuous. With f so defined we may write

$$Lp^2w + Rpw + Sw = Rf. \tag{4}$$

LEMMA 5. *Let* $G = (zL + R + z^{-1}S)^{-1}R$; *then* G, zLG *and* $z^{-1}SG$ *are uniformly bounded matrices for* $\Re z \geqq 0$.

PROOF. First suppose z is real and $1 \leqq z < \infty$. Let h be an arbitrary vector and let $x = Gh$. Then

$$(zL + R + z^{-1}S)x = Rh$$

and

$$z\{x, Lx\} + \{x, Rx\} + z^{-1}\{x, Sx\} = \{x, Rh\}.$$

Apply Lemma 2 to each of the terms on the left side. On the right side note $\{x, Rh\} = \{h, Rx\} \leqq \|h\| \, \|Rx\|$. Thus $z\|Lx\|^2 + \|Rx\|^2 + z^{-1}\|Sx\|^2 \leqq A\|h\| \, \|Rx\|$. This makes it obvious that $\|Rx\|$ is bounded, hence $\|Lx\|$ is bounded and $\|Sx\| = Oz^{1/2}$. Thus $\|(L+R+S)x\| = Oz^{1/2}$. But $(L+R+S)$ is a nonsingular constant matrix, so actually $\|x\| = Oz^{1/2}$. Since h is an arbitrary vector the matrix elements of G, say g_{ij}, must satisfy $|g_{ij}| = Oz^{1/2}$. But g_{ij} is a rational function of z and if it were unbounded it would increase at least as rapidly as z. Thus g_{ij} is bounded at infinity. To show g_{ij} is bounded in the neighborhood of the origin, let $z' = 1/z$ and repeat the above argument. Since g_{ij} has no singularities for $\Re z \geqq 0$ and $|z| > 0$, it follows that g_{ij} is uniformly bounded for $\Re z \geqq 0$. This proves the statement for G.

By what has just been proved, zLG is bounded in the neighborhood of the origin. Write

$$\begin{aligned} zLG &= -RG - z^{-1}SG + (zL + R + z^{-1}S)G \\ &= -RG - z^{-1}SG + R. \end{aligned}$$

The three matrices on the right are uniformly bounded at infinity and this proves the statement for zLG. The remaining part follows by symmetry.

LEMMA 6. *Equation* (4) *has a continuous solution* $pw_1 \subset L_2(0, \infty)$.

PROOF. Let $\phi(z) = \int_0^\infty e^{-zt} f(t)dt$, $\Re z > 0$. The existence of $\phi(z)$ is guaranteed since we have shown $f \subset L_2(0, \infty)$. For brevity write $\phi(z) = \mathfrak{L}f(t)$. Let $\zeta(z) = G\phi(z)$, and since G is a uniformly bounded rational transformation for $\Re z \geqq 0$ it follows from the well known theory of the Laplace transformation that $\zeta(z) = \mathfrak{L}u(t)$, where $u(t) \subset L_2(0, \infty)$ and is continuous. Moreover $z^{-1}\zeta(z) = \mathfrak{L}\int_0^t u dt$ and $z^{-1}S\zeta(z) = \mathfrak{L}S\int_0^t u dt$. Likewise by Lemma 5, $zL\zeta(z) = \mathfrak{L}l(t)$, where $l(t) \subset L_2(0, \infty)$ and is continuous. Thus $\mathfrak{L}\int_0^t l dt = \mathfrak{L}Lu$. By the uniqueness property of the Laplace transform, $\int_0^t l dt = Lu$, so $l = Ldu/dt$. Note that

$$\mathfrak{L}\left(Ldu/dt + Ru + S\int_0^t u dt\right) = (zL + R + z^{-1}S)\zeta = \mathfrak{L}Rf.$$

Thus $Ldu/dt + Ru + S\int_0^t u dt = Rf$. Identifying w_1 with $\int_0^t u dt$ completes the proof.

Clearly $x = w - w_1$ satisfies the linear homogeneous equation and px is continuous so $px \rightarrow 0$ as $t \rightarrow +\infty$ by hypothesis. Moreover, it is easy to show by the Laplace transformation that $px \subset L_2(0, \infty)$. Thus $w \subset L_2(0, \infty)$ and the proof is completed that pq and pq^* approach each other in mean. We have been unable to show that the solutions also approach in the pointwise sense.

Under the weaker hypothesis that px merely approaches a constant the same proof shows that pw also approaches a constant.

A later note considers the existence and character of solutions of quasi-linear networks for periodic impressed force.

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THE ELECTROSTATIC FIELD OF TWO COPLANAR PLATES

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1. **Introduction.** In a recent paper in the Philosophical Magazine [1, p. 168],[1] N. Davy published what he called an "attempt" to obtain the electrostatic field about two thin, infinitely long, parallel, coplanar metallic plates of unequal width and at potentials $\pm V_0$. It was this remark, no doubt, that led M. C. Gray [2] to call this solution "tentative."

Actually, the solution given by Davy was but one of infinitely many that might fit his given conditions. The reason is that this is a two-dimensional potential problem, and in two-dimensional potential theory infinity is not a suitable zero point for the potential function as it is in three-dimensional potential theory. Consequently, to make the potential function definite, it is necessary either to specify its zero point or to specify some other condition which effectively does this. Davy made no such specification but chose the particular potential function which corresponds to the case in which the charges on the conductors are equal and opposite in sign.

It is the purpose of this paper to solve the problem fulfilling Davy's conditions, but in which the charge per unit length on one plate bears to the charge per unit length on the other plate a given ratio r.

For any $r \neq 1$, the charges on the plates may be increased until the potential difference between the two plates is $2V_0$. If the zero point for the potential function is then taken as the point between the two plates at which the potential is the arithmetic mean of the potentials on the plates, one plate will be at potential $+V_0$ and the other will

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[1] Numbers in brackets refer to the references cited at the end of the paper.