ON THE IRREDUCIBILITY OF CERTAIN POLYNOMIALS

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Introduction. G. Pólya has proved the following theorem:
If for \( n \) integral values of \( x \), the integral polynomial \( P(x) \) of degree \( n \) has values which are different from zero and, without regard to sign, less than
\[
G_1 = \frac{(n - \lfloor n/2 \rfloor)!}{2^{n-\lfloor n/2 \rfloor}},
\]
then \( P(x) \) is irreducible in the field of rational numbers. (Here, as in the following, a polynomial with rational integral coefficients is called an "integral polynomial").

This result was improved for positively definite polynomials by Hildegard Ille and for arbitrary polynomials by T. Tatuzawa. The latter obtained the larger bound
\[
G_2 = (2^{-n}(n - 1)!)^{1/2}
\]
instead of \( G_1 \). Moreover he proved the following theorem:
If for \( l \) integral values of \( x \) where \( n > l > n/2 \), the integral polynomial \( P(x) \) of degree \( n \) takes values which are different from zero and, without regard to sign, less than
\[
H_1 = (l - 1)!^{1/2},
\]
then \( P(x) \) is irreducible in the field of rational numbers.

In the following, the results of Tatuzawa will be improved further by a slight modification of his method. Instead of \( G_2 \) we obtain the larger bound
\[
G = \frac{(n - 1)!}{2^{n-1}(n - 2)/2}.
\]

We have
\[
\frac{G}{G_2} \sim \left(\frac{2n}{\pi}\right)^{1/4} \text{ for even } \ n \text{ and } \frac{G}{G_2} \sim \left(\frac{n^8}{2\pi}\right)^{1/4} \text{ for odd } \ n.
\]
It follows from this theorem in particular that the integral polynomials
\[ P(x) = A(x - x_1)(x - x_2) \cdots (x - x_n) + t \]
are irreducible if \( x_n \neq x_\lambda \) and \( 1 \leq |t| < G \). This result contains for \( n > 4 \), \( A = 1 \), and \( t = \pm 1 \) a theorem of I. Schur,\(^4\) and for arbitrary \( A \) and \( t = \pm 1 \) a theorem of H. L. Dorwart and O. Ore.\(^6\) For \( n \leq 4 \) there are exceptions.\(^6\) The application of Pólya's bound gives this result only for \( n > 6 \), that of Tatużawa for \( n > 5 \), while our bound gives the exact degrees for which these theorems hold.

Moreover, we improve the second theorem of Tatużawa. Instead of \( H_1 \) we obtain the larger bound
\[ H = \left[ \frac{l + 1}{2} \right] \text{ for } l \geq 7, \quad H = \frac{3}{2} \text{ for } l = 6 \text{ and } 5. \]

It is of interest that this bound cannot be improved further. If the absolute value of \( P(x_\lambda) \) for \( \lambda = 1, 2, \ldots, 1 \) is not less than \( H \), but only less than or equal to \( H \), then our result does not remain correct. It will be shown that for every \( n > 2 \) such polynomials exist which are reducible in the field of rational numbers.

It follows from our theorem that the integral polynomials of degree \( n \)
\[ P(x) = (x - x_1)(x - x_2) \cdots (x - x_l)h(x) + t \]
are irreducible if \( l > 4 \), \( n > l > n/2 \), \( x_\lambda \neq x_n \), and \( 1 \leq |t| < H \). This gives for \( l = \pm 1 \) a theorem of Dorwart and Ore.\(^7\)

Finally a criterion of a new type is obtained. \( P(x) \) is irreducible if the absolute value of \( P(x_\lambda) \) is different from zero and less than a certain bound \( S_1 > G \) for \( n \) different integers \( x_\lambda \), but less than another smaller constant \( S_2 \) for \( l \) of these \( x_\lambda \). More exactly, the following theorem is proved.

Let \( P(x) \) be an integral polynomial of degree \( n \); let \( k, l, \) and \( h \) be integers satisfying the following conditions: \( k \geq \lceil (n+1)/2 \rceil \), \( n > l > n/2 \), \( l > 12 \) or \( l = 11 \) or \( 9 \),
\[ l > h, \quad n \geq k + h - 1. \]

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\(^7\) See, for instance, loc. cit. footnote 5, pp. 86, 195.
If for \( n \) different integers \( x_1, x_2, \ldots, x_n \)
\[
0 < |P(x)| < 2^{-k} \prod_{\nu=1}^{k} \left[ \frac{\nu - 1}{k} \right] \quad (\nu = 1, 2, \ldots, n),
\]
and for \( l \) of these \( x_i \), say \( x^{(1)}, x^{(2)}, \ldots, x^{(l)} \)
\[
|P(x^{(\lambda)})| < 2^{-k} \prod_{\mu=1}^{k} \left[ \frac{\mu - 1}{h} \right] \quad (\lambda = 1, 2, \ldots, l),
\]
then \( P(x) \) is irreducible in the field of rational numbers.

1. Bounds for the absolute value of polynomials at given points.

We first prove the following theorem of Tatuzawa.

**Theorem 1.** Let \( f(x) = a_0x^k + a_1x^{k-1} + \cdots + a_k \) be a polynomial of degree \( k \), let \( x_1 < x_2 < \cdots < x_{k+1} \) be arbitrary numbers, and \( d \) the length of the smallest interval which contains \( k+1 \) of these numbers \( (k = 1, 2, \ldots, k) \). Then
\[
\max_{\kappa=1,2,\ldots,k+1} |f(x_\kappa)| \geq 2^{-k} |a_0| d_1 d_2 \cdots d_k.
\]

**Proof.** For \( k = 1 \) we have
\[
|f(x_1) - f(x_2)| = |a_0(x_1 - x_2)| = |a_0| d_1,
\]
hence
\[
\max \{ |f(x_1)|, |f(x_2)| \} \geq 2^{-1} \{ |f(x_1)| + |f(x_2)| \} \\
= 2^{-1} |f(x_1) - f(x_2)| = 2^{-1} |a_0| d_1.
\]
This proves the theorem for \( k = 1 \). Let us now assume that it is already proved for polynomials of degree less than \( k \).

We divide \( f(x) \) by the polynomials \( (x-x_1)(x-x_2) \cdots (x-x_k) \) and \( (x-x_2)(x-x_3) \cdots (x-x_{k+1}) \), respectively. Then
\[
f(x) = a_0(x-x_1)(x-x_2) \cdots (x-x_k) + g(x) \\
= a_0(x-x_2)(x-x_3) \cdots (x-x_{k+1}) + h(x)
\]
where \( g(x) \) and \( h(x) \) are polynomials of degree less than \( k \) with the highest coefficients
\[
b_0 = a_1 + a_0(x_1 + x_2 + \cdots + x_k)
\]
and
\[
c_0 = a_1 + a_0(x_2 + x_3 + \cdots + x_{k+1}),
\]
respectively, hence \( b_0 - c_0 = a_0(x_1 - x_{k+1}) \).
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\[ \max (|b_0|, |c_0|) \geq 2^{-1}(|b_0| + |c_0|) \]
\[ \geq 2^{-1} |b_0 - c_0| = 2^{-1} |a_0| d_k. \]

On the other hand it follows from (1) that
\[ \max_{\kappa=1, 2, \cdots, k+1} |f(x_{\kappa})| \geq \max_{\kappa=1, 2, \cdots, k} |g(x_{\kappa})|, \]
\[ \max_{\kappa=1, 2, \cdots, k+1} |f(x_{\kappa})| \geq \max_{\kappa=2, 3, \cdots, k+1} |h(x_{\kappa})|. \]

Since \( g(x) \) and \( h(x) \) are of lower degree than \( k \), our theorem may be applied to them. The lengths of the smallest intervals which contain \( \kappa \) of the points \( x_1, x_2, \cdots, x_k \) or \( \kappa \) of the points \( x_2, x_3, \cdots, x_{k+1} \) are both not smaller than \( d_{k-1} \). Hence
\[ \max_{\kappa=1, 2, \cdots, k} |g(x_{\kappa})| \geq 2^{-k+1} |b_0| d_1 d_2 \cdots d_{k-1}, \]
\[ \max_{\kappa=2, 3, \cdots, k+1} |h(x_{\kappa})| \geq 2^{-k+1} |c_0| d_1 d_2 \cdots d_{k-1}, \]
and by (3) and (2)
\[ \max_{\kappa=1, 2, \cdots, k+1} |f(x_{\kappa})| \geq 2^{-k+1} d_1 d_2 \cdots d_{k-1} \left\{ \max \left( |b_0|, |c_0| \right) \right\} \]
\[ = 2^{-k} |a_0| d_1 d_2 \cdots d_k. \]

\textbf{THEOREM 2.} Let \( f(x) = a_0 x^k + a_1 x^{k-1} + \cdots + a_h \) be a polynomial of degree \( k \) and \( x_1 < x_2 < \cdots < x_n \) a set of more than \( k \) integers. Then
\[ \max_{\mu=1, 2, \cdots, n} |f(x_{\mu})| \geq 2^{-k} |a_0| \prod_{\nu=1}^{k} \left[ \frac{\kappa \nu - 1}{k} \right]. \]

\textbf{PROOF.} We consider those \( k+1 \) of the integers \( x_{\kappa} \), whose subscripts are 1 and the \( k \) numbers
\[ 1 + \left\lfloor \frac{\rho n - 1}{k} \right\rfloor (\rho = 1, 2, \cdots, k), \]
and denote these \( x_{\kappa} \), in increasing order by \( x_{\beta}', x_{\beta}', \cdots, x_{\alpha}' \). The difference of two consecutive elements \( x_{\alpha}' \) is at least 1, hence for \( k \geq \beta > \alpha > 0 \)
\[ x_{\beta}' - x_{\alpha}' \geq \left\lfloor \frac{\beta n - 1}{k} \right\rfloor - \left\lfloor \frac{\alpha n - 1}{k} \right\rfloor. \]

Since for every \( r \) and \( s \)
\[ \left\lfloor \frac{r + s}{k} \right\rfloor = \left\lfloor \frac{r}{k} \right\rfloor + \left\lfloor \frac{s}{k} \right\rfloor, \]
we have
hence by (4)
\[ x^\prime_\beta - x^\prime_\alpha \leq \left( \frac{(\beta - \alpha)n - 1}{k} \right) + \left[ \frac{\alpha n}{k} \right] - \left[ \frac{\alpha n - 1}{k} \right] \]
\[ \leq \left( \frac{(\beta - \alpha)n - 1}{k} \right). \tag{5} \]

It is obvious that (5) holds also for \( \alpha = 0 \). If the length of the smallest interval which contains \( \kappa + 1 \) of the numbers \( x^\prime_1, x^\prime_2, \ldots, x^\prime_\kappa \) is denoted by \( d^\prime_\kappa \), then by (5)
\[ d^\prime_\kappa \geq \left[ \frac{\kappa n - 1}{k} \right] \]

hence, by Theorem 1,
\[
\max_{\nu=1,2,\ldots,n} |f(x^\nu)| \geq \max_{\kappa=0,1,\ldots,k} |f(x^\nu_\kappa)| \geq 2^{-k} \cdot a_0 \cdot \prod_{\kappa=1}^{k} \left[ \frac{\kappa n - 1}{k} \right].
\]

2. Criteria for irreducibility. Theorem 2 will be used now to obtain criteria for irreducibility of polynomials.

Theorem 3. Let \( P(x) \) be a polynomial of degree \( n \) with integral rational coefficients. If for \( n \) integral values \( x_1, x_2, \ldots, x_n \) the absolute value of \( P(x^\nu) \) for \( \nu = 1, 2, \ldots, n \) is less than

\[ G = \frac{(n - 1)!}{2^{n-1} \cdot \left( (n - 2)/2 \right)!}, \]

but different from 0, then \( P(x) \) is irreducible in the field of rational numbers.

Proof. If \( P(x) \) is reducible, then it contains a factor \( f(x) \) of degree \( k \) with integral coefficients where \( n > k \geq \lceil (n + 1)/2 \rceil \). It follows now from Theorem 2 that

\[ M = \max_{\nu=1,2,\ldots,n} |f(x^\nu)| \geq 2^{-k} \cdot \prod_{\kappa=1}^{k} \left[ \frac{\kappa n - 1}{k} \right]. \tag{6} \]

We set for fixed \( n \)
\[ \phi(k) = \phi(k, n) = 2^{-k} \cdot \prod_{\kappa=1}^{k} \left[ \frac{\kappa n - 1}{k} \right]. \tag{7} \]

Let us first assume that \( k = \lceil (n + 1)/2 \rceil \). For even values of \( n \) we have
\[
\left[ \frac{\kappa n - 1}{k} \right] = \left[ \frac{\kappa n - 1}{n/2} \right] = \left[ \frac{2\kappa n - 2n}{n} \right] = 2\kappa - 1 \quad (\kappa = 1, 2, \ldots, k),
\]

hence by (7)

\[
\phi\{\left[ (n + 1)/2 \right]\} \geq 2^{-k} \prod_{\kappa=1}^{k} (2\kappa - 1) = 2^{-n/2} \cdot 1 \cdot 3 \cdot 5 \cdots (n - 1)
\]

(8)

\[
= \frac{(n - 1)!}{2^{n/2 + (n-3)/2} \left( (n - 2)/2 \right)!} = \frac{(n - 1)!}{2^{n-1} \left\{ \left( (n - 2)/2 \right)! \right\}} = G.
\]

If \( n \) is odd, then

\[
\left[ \frac{\kappa n - 1}{k} \right] = \left[ \frac{\kappa n - 1}{(n + 1)/2} \right] = \left[ \frac{2\kappa n - 2}{n + 1} \right]
\]

\[
= \left[ \frac{(2\kappa - 1)(n + 1) + n - 2\kappa - 1}{n + 1} \right],
\]

hence

\[
\left[ \frac{\kappa n - 1}{k} \right] = \begin{cases} 2\kappa - 1 & \text{for } \kappa = 1, 2, \ldots, (n - 1)/2, \\ 2\kappa - 2 & \text{for } \kappa = (n + 1)/2, \end{cases}
\]

and by (7)

\[
\phi\{\left[ (n + 1)/2 \right]\} \geq 2^{-(n+1)/2} \left( n - 1 \right) \prod_{\kappa=1}^{(n-1)/2} (2\kappa - 1)
\]

(9)

\[
= \frac{(n - 1)!}{2^{(n+1)/2 + (n-3)/2} \left\{ \left( (n - 3)/2 \right)! \right\}} = \frac{(n - 1)!}{2^{n-1} \left\{ \left( (n - 2)/2 \right)! \right\}} = G.
\]

It follows from (8) and (9) that

\[
\phi\left( \left[ \frac{n + 1}{2} \right] \right) \geq G.
\]

(10)

Now we maintain that

\[
\phi(k + 1) \geq \phi(k) \quad \text{for} \quad \left[ \frac{n + 1}{2} \right] \leq k \leq n - 2.
\]

(11)

For this purpose we want to prove that

\[
\left[ \frac{(\kappa + 1)n - 1}{k + 1} \right] \geq \left[ \frac{\kappa n - 1}{k} \right] \quad (\kappa = 2, 3, \ldots, k).
\]

(12)

We divide \( \kappa n - 1 \) by \( k \):
\[ kn - 1 = qk + r \quad (0 \leq r < k), \]

hence \( kn > qk \) and \( n > q \) since \( k \leq k \). It follows that

\[
\left( \frac{(k + 1)n - 1}{k + 1} \right) = \left( \frac{qk + r + n}{k + 1} \right) = \left( \frac{q(k + 1) + r + n - q}{k + 1} \right)
\]

\[ \geq q = \left\lfloor \frac{kn - 1}{k} \right\rfloor. \]

This proves (12). Moreover we have for \( \kappa = 1 \)

\[
\left( \frac{2n - 1}{k + 1} \right) \geq \left( \frac{2n - 1}{n - 1} \right) = 2 = 2 \left\lfloor \frac{n - 1}{k} \right\rfloor
\]

because it is sufficient for the proof of (11) to assume that \( k + 1 \leq n - 1 \) and \( k \geq \left\lfloor \frac{(n+1)/2}{k} \right\rfloor \). Multiplying (13) and (12) for \( \kappa = 2, 3, \ldots, k \) we obtain

\[
\prod_{\kappa = 1}^{k} \left( \frac{(k + 1)n - 1}{k + 1} \right) \geq 2 \prod_{\kappa = 1}^{k} \left\lfloor \frac{kn - 1}{k} \right\rfloor,
\]

hence, since \( \left\lfloor \frac{(n-1)/(k+1)}{1} \right\rfloor = 1, \)

\[
2^{-k} \prod_{\kappa = 1}^{k+1} \left( \frac{kn - 1}{k + 1} \right) = 2^{-k} \left( \frac{n - 1}{k + 1} \right) \prod_{\kappa = 1}^{k} \left( \frac{(k + 1)n - 1}{k + 1} \right)
\]

\[ \geq 2^{-k} \prod_{\kappa = 1}^{k} \left( \frac{kn - 1}{k} \right). \]

This proves (11). It follows now from (6), (7), (11), and (10) that

\[ M \geq G. \]

Since \( P(x)/f(x) \) is a polynomial with integral coefficients and \( P(x) \neq 0 \), we obtain

\[ \max_{r=1, 2, \ldots, n} |P(x_r)| \geq \max_{r=1, 2, \ldots, n} |f(x_r)| = M \geq G. \]

This contradicts our assumption, and the theorem is proved.

The bound of Tatuwa is

\[ G_2 = (2^{-n(n - 1)}/1)^{1/2} \]

and our bound by (8) and (9)

\[
G = \frac{(n - 1)!}{2^{n-1} \left\lfloor [(n - 2)/2]! \right\rfloor} = \begin{cases} 2^{-n/2} \cdot 1 \cdot 3 \cdot 5 \cdots (n - 1) & \text{for even } n, \\ 2^{-(n+1)/2} (n - 1) \cdot 1 \cdot 3 \cdot 5 \cdots (n - 2) & \text{for odd } n. \end{cases}
\]
Now we have by the formula of Wallis
\[
\frac{1 \cdot 3 \cdot 5 \cdots (2m - 1)}{2 \cdot 4 \cdot 6 \cdots (2m - 2)} \sim \left(\frac{4m}{\pi}\right)^{1/2}.
\]
For even \(n\) we obtain
\[
\frac{G}{G_2} = \left(\frac{1 \cdot 3 \cdot 5 \cdots (n - 1)}{2 \cdot 4 \cdot 6 \cdots (n - 2)}\right)^{1/2} \sim \left(\frac{2n}{\pi}\right)^{1/4},
\]
and for odd \(n\)
\[
\frac{G}{G_2} = \frac{1}{2^{1/2}} \left(\frac{(n - 1) \cdot 1 \cdot 3 \cdot 5 \cdots (n - 2)}{2 \cdot 4 \cdot 6 \cdots (n - 3)}\right)^{1/2} \sim \left(\frac{n}{2}\right)^{1/2} \left(\frac{2n}{\pi}\right)^{1/4} = \left(\frac{n^5}{2\pi}\right)^{1/4}.
\]
Moreover, we have \(G > G_2\) for \(n > 3\).

**COROLLARY.** The integral polynomials
\[
(14) \quad A(x - x_1)(x - x_2) \cdots (x - x_n) + t
\]
are irreducible in the field of rational numbers if \(x_i \neq x_j\) and \(1 \leq |t| < G\).

We have \(G > 1\) for \(n > 4\). Therefore the polynomials (14) are irreducible for \(t = \pm 1\) and \(n > 4\). This is, as already mentioned in the introduction, for \(A = 1\) a theorem of Schur, and for arbitrary \(A\) a theorem of Dorwart and Ore.

We can formulate Theorem 3 also in the following form:

**THEOREM 3a.** Let \(P(x)\) be an integral polynomial of degree \(n\). If for \(n\) different integers \(x_1, x_2, \cdots, x_n\)
\[
0 < |P(x)| < 2^{-k} \prod_{r=1}^{k} \left\lfloor \frac{kn - 1}{k} \right\rfloor = \phi(k) \quad (\nu = 1, 2, \cdots, n)
\]
where \(k \geq \lceil (n+1)/2 \rceil\), then \(P(x)\) cannot contain a factor \(f^*(x)\) of degree \(k^*\) with \(k \leq k^* < n\).

**PROOF.** If \(P(x)\) contains such a factor, then by (6), (7), and (11)
\[
\max_{r=1,2,\cdots,n} |P(x_r)| \geq \max_{r=1,2,\cdots,n} |f^*(x_r)| \geq \phi(k^*) \geq \phi(k).
\]
This contradicts our assumption.

**THEOREM 4.** Let \(P(x)\) be a polynomial of degree \(n\) with integral coefficients, \(l\) an integer with \(l \geq 5\), and \(n > l > n/2\). If for \(l\) different integers \(x_1, x_2, \cdots, x_l\)
\[
0 < |P(x)| < H \quad (\lambda = 1, 2, \cdots, l)
\]
where $H = [(l+1)/2]$ for $l \geq 7$, $H = 3/2$ for $l = 6$ and $5$, then $P(x)$ is irreducible in the field of rational numbers.

**Proof.** If $P(x)$ is reducible, then it must contain a factor $g(x)$ of degree $k$ less than or equal to $n/2$ with integral coefficients. Here $k > 1$. For a linear polynomial takes each value only once, hence the $l$ integers $g(x_1), g(x_2), \ldots, g(x_l)$ must be different. However, only the $2 [(l-1)/2]$ values $\pm 1, \pm 2, \ldots, \pm [(l-1)/2]$ are possible because

$$0 < |g(x_\lambda)| \leq |P(x_\lambda)| < H \leq \left[\frac{l+1}{2}\right]$$

($\lambda = 1, 2, \ldots, l$).

This gives a contradiction since $l > 2 [(l-1)/2]$.

It follows from Theorem 2 that

$$\max_{\lambda=1,2,\ldots,l} |g(x_\lambda)| \geq 2^{-k} \prod_{k=1}^{l} \left[\frac{\kappa l - 1}{k}\right].$$

We denote the right-hand side of (15), for a given $l$, similarly as in (10), by $\phi(k)$ and maintain that

$$\phi(k+1) \geq \phi(k) \quad \text{for} \quad 2 \leq k \leq (l-3)/2 \quad \text{and for} \quad l/2 \leq k \leq l-2.$$  

It follows from the proof of Theorem 3 that (16) holds for $l/2 \leq k \leq l-2$ if we write $l$ instead of $n$ since $[(l+1)/2] = l/2$ for even $l$ and $k = l/2$ for odd $l$.

Now we consider the case $k \leq (l-3)/2$. Here we have

$$\left[\frac{l-1}{k+1}\right] \geq \left[\frac{l-1}{(l-1)/2}\right] = 2.$$  

Moreover, it follows from (12) that

$$\left[\frac{(\kappa + 1)l - 1}{k+1}\right] \geq \left[\frac{\kappa l - 1}{k}\right] \quad \text{for} \quad \kappa = 1, 2, \ldots, k.$$  

We proved (12) only for $\kappa \geq 2$; but the proof remains correct for $\kappa = 1$. It follows now from (18) and (17) that

$$2^{-k-1} \prod_{k=0}^{l} \left[\frac{(\kappa + 1)l - 1}{k+1}\right] = 2^{-k-1} \prod_{k=1}^{l} \left[\frac{\kappa l - 1}{k+1}\right] \geq 2^{-k} \prod_{k=1}^{l} \left[\frac{\kappa l - 1}{k}\right],$$

hence $\phi(k+1) \geq \phi(k)$ for $k \leq (l-3)/2$, and (16) is proved.

We now consider the remaining value $k = [(l-1)/2]$. We maintain that also here
(19) \( \phi(k + 1) \geq \phi(k) \) for \( l > 12 \) and \( l = 11, 9 \).

For even \( l \) we have \( k = (l - 2)/2 \). It follows from (7) and (8) that for \( l > 12 \)

\[
\phi(k + 1) = \phi \left( \frac{l + 1}{2} \right) \geq 2^{-\frac{l}{2}} \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot \prod_{k=5}^{k+1} \frac{k\ell - 1}{k + 1}
\]

(20) 

\[
= 2^{-\frac{l}{2}} \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot \prod_{k=4}^{k} \frac{(k + 1)l - 1}{k + 1}
\]

and

\[
\phi(k) = \phi \left( \frac{l - 1}{2} \right)
\]

\[
= 2^{-(l-2)/2} \left[ \frac{l - 1}{l - 2} \right] \left[ \frac{2l - 1}{l - 2} \right] \left[ \frac{3l - 1}{l - 2} \right] \prod_{k=4}^{k} \left[ \frac{k\ell - 1}{k} \right]
\]

\[
= 2^{-(l-2)/2} \left[ \frac{2l - 2}{l - 2} \right] \left[ \frac{4l - 2}{l - 2} \right] \left[ \frac{6l - 2}{l - 2} \right] \prod_{k=4}^{k} \left[ \frac{k\ell - 1}{k} \right].
\]

Now \( [(6l - 2)/(l - 2)] = 6 \) since \( 6l - 2 < (l - 2) \) for \( l > 12 \). Similarly \( [(2l - 2)/(l - 2)] = 2 \) and \( [(4l - 2)/(l - 2)] = 4 \), hence

(21) 

\[
\phi(k) \geq 2^{-(l-2)/2} \cdot 2 \cdot 4 \cdot 6 \cdot \prod_{k=4}^{k} \left[ \frac{k\ell - 1}{k} \right].
\]

Since \( 3 \cdot 5 \cdot 7 > 2 \cdot 2 \cdot 4 \cdot 6 \), it follows from (20), (21), and (12) that (19) holds for even \( l > 12 \).

Now, let \( l \) be odd. Here we have by (7) and (9) for \( l > 7 \)

\[
\phi(k + 1) = \phi \left( \frac{l + 1}{2} \right) = 2^{-(l+1)/2} \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot \prod_{k=5}^{k+1} \frac{k\ell - 1}{k + 1}
\]

(22) 

\[
= 2^{-(l+1)/2} \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot \prod_{k=4}^{k} \frac{(k + 1)l - 1}{k + 1}
\]

and

\[
\phi(k) = \phi \left( \frac{l - 1}{2} \right)
\]

\[
= 2^{-(l-1)/2} \left[ \frac{2l - 2}{l - 1} \right] \left[ \frac{4l - 2}{l - 1} \right] \left[ \frac{6l - 2}{l - 1} \right] \prod_{k=4}^{k} \left[ \frac{k\ell - 1}{k} \right]
\]

(23) 

\[
= 2^{-(l-1)/2} \cdot 2 \cdot 4 \cdot 6 \cdot \prod_{k=4}^{k} \left[ \frac{k\ell - 1}{k} \right]
\]

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since $6l - 2 < 7(l - 1)$. It follows from (22), (23), and (12) that (19) holds for odd $l > 7$, too.

Now we have by (16) and (19)

$\phi(k + 1) \geq \phi(k)$ for $2 \leq k \leq l - 2$, and $l > 12, l = 11, 9$.

By direct calculation we obtain

$\phi(5) > \phi(6) > \phi(4)$ for $l = 12$,

$\phi(4) > \phi(5) > \phi(3)$ for $l = 10$,

$\phi(3) > \phi(4) > \phi(2)$ for $l = 8$ and $l = 7$.

By (24) and (25) we have for $l \geq 7$

$\phi(k) \geq \phi(2) = \left[ \frac{2l - 1}{2} \right] = H$ (3 \leq k < l),

and by direct calculation

$\phi(k) \geq 3/2 = H$ for $l = 6$ and 5 (2 \leq k < l),

hence, by (15), (26), and (27),

$max_{\lambda=1,2,\ldots,l} |P(x_\lambda)| \geq max_{\lambda=1,2,\ldots,l} |g(x_\lambda)| \geq \phi(k) \geq H$.

This contradicts our assumption, and $P(x)$ must be irreducible.

Theorem 4 cannot be improved further. If we assume instead of

$0 < |P(x)| < \left[ (l + 1)/2 \right]$ (X= 1, 2, \ldots, l)

only

$0 < |P(x)| \leq \left[ (l + 1)/2 \right]$ (X= 1, 2, \ldots, l),

then $P(x)$ may be reducible.

This is shown by the following examples

$P(x) = x \left\{ h(x) \prod_{\lambda=1}^{(l-1)/2} (x^2 - \lambda^2) + 1 \right\}$ for even $l$,

$P(x) = x \left\{ h(x) \left( x - \frac{l + 1}{2} \right)^{(l-1)/2} \prod_{\lambda=1}^{(l-1)/2} (x^2 - \lambda^2) + 1 \right\}$ for odd $l$,

where $h(x)$ is an arbitrary integral polynomial of degree $n - l - 1$. We have here $P(x) = x$ for $x = \pm 1, \pm 2, \ldots, \pm l/2$ for even $l$, and $x = \pm 1, \pm 2, \ldots, \pm (l - 1)/2, \pm (l + 1)/2$ for odd $l$. At $l$ integral points these polynomials take values which are different from zero and, without regard to sign, less than or equal to $[(l + 1)/2]$; but they are reducible.
COROLLARY. Let \( h(x) \) be an arbitrary integral polynomial of degree \( n-1 \) and \( x_1, x_2, \ldots, x_1 \) different integers. The integral polynomial \( P(x) \) of degree \( n \),

\[
P(x) = (x - x_1)(x - x_2) \cdots (x - x_1)h(x) + t,
\]
is irreducible in the field of rational numbers if \( l \geq 5, n > l > n/2 \) and

\[
1 \leq |t| < H \text{ where } H = [(l+1)/2] \text{ for } l \geq 7 \text{ and } H = 3/2 \text{ for } l = 6 \text{ and } 5.
\]

For the proof of Theorem 4 for \( l > 12 \) it is sufficient to prove only (26) instead of (24). We proved here (24) in order to obtain the following theorem.

THEOREM 4a. Let \( P(x) \) be an integral polynomial of degree \( n \); let \( h \) and \( l \) be integers satisfying the following conditions:

\[
n > l > n/2, \quad l > h \geq 2, \quad \text{and } l > 12, \text{ or } l = 11, 9.
\]

If for \( l \) different integers \( x_1, x_2, \ldots, x_1 \)

\[
0 < |P(x_\lambda)| < \phi(h) = 2^{-k} \prod_{\mu=1}^{h} \left[\frac{\mu l - 1}{h}\right],
\]

then \( P(x) \) cannot contain a factor of degree \( h^* \) with \( h \leq h^* \leq n/2 \).

PROOF. If \( P(x) \) contains a factor of degree \( h^* \), then by (28) and (24)

\[
\max_{\lambda=1, 2, \ldots, l} |P(x_\lambda)| \geq \max_{\lambda=1, 2, \ldots, l} |g(x_\lambda)| \geq \phi(h^*) \geq \phi(h).
\]

This contradicts our assumption.

A similar theorem can be obtained for \( l = 12, 10, 8, \) and 7.

By combining Theorems 3a and 4a the following theorem is obtained.

THEOREM 5. Let \( P(x) \) be an integral polynomial of degree \( n \), and \( k, l, \) and \( h \) integers satisfying the following conditions:

\[
k \geq [(n + 1)/2], \quad n > l > n/2, \quad l > 12 \text{ or } l = 11 \text{ or } 9,
\]

\[
l > h, \quad n \geq k + h - 1.
\]

If for \( n \) different integers \( x_1, x_2, \ldots, x_n \)

\[
0 < |P(x_\nu)| < 2^{-k} \prod_{\kappa=1}^{k} \left[\frac{\nu n - 1}{k}\right] \quad (\nu = 1, 2, \ldots, n),
\]

and for \( l \) of these \( x_\nu \), say \( x^{(1)}, x^{(2)}, \ldots, x^{(l)} \),
\[ |P(x^{(\lambda)})| < 2^{-h} \prod_{\mu=1}^{h} \left[ \frac{\mu l - 1}{h} \right] \quad (\lambda = 1, 2, \ldots, l), \]

then \( P(x) \) is irreducible in the field of rational numbers.

**Proof.** If \( P(x) \) is reducible, then

\[ P(x) = f(x) \cdot g(x). \]

If the degree of \( f(x) \) is \( k^* \geq n/2 \), then the degree of \( g(x) \) is \( n - k^* \). It follows from Theorem 3a that \( k^* \leq k - 1 \), and from Theorem 4a that \( n - k^* \leq h - 1 \), hence \( n \leq k + h - 2 \). This gives a contradiction.